# OPTOELECTRONICS (I)

# Chapter 4-2:
# II. Cylindrical Waveguides

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# 4-1 Acceptance Angle and Numerical Aperture

Total internal reflection will occur at the core-cladding boundary if $n_2 < n_1$, *provided that the internal waveguide angle* $\theta$ *is* **greater** than the **critical angle** $\theta_c = \sin^{-1}(n_2/n_1)$.

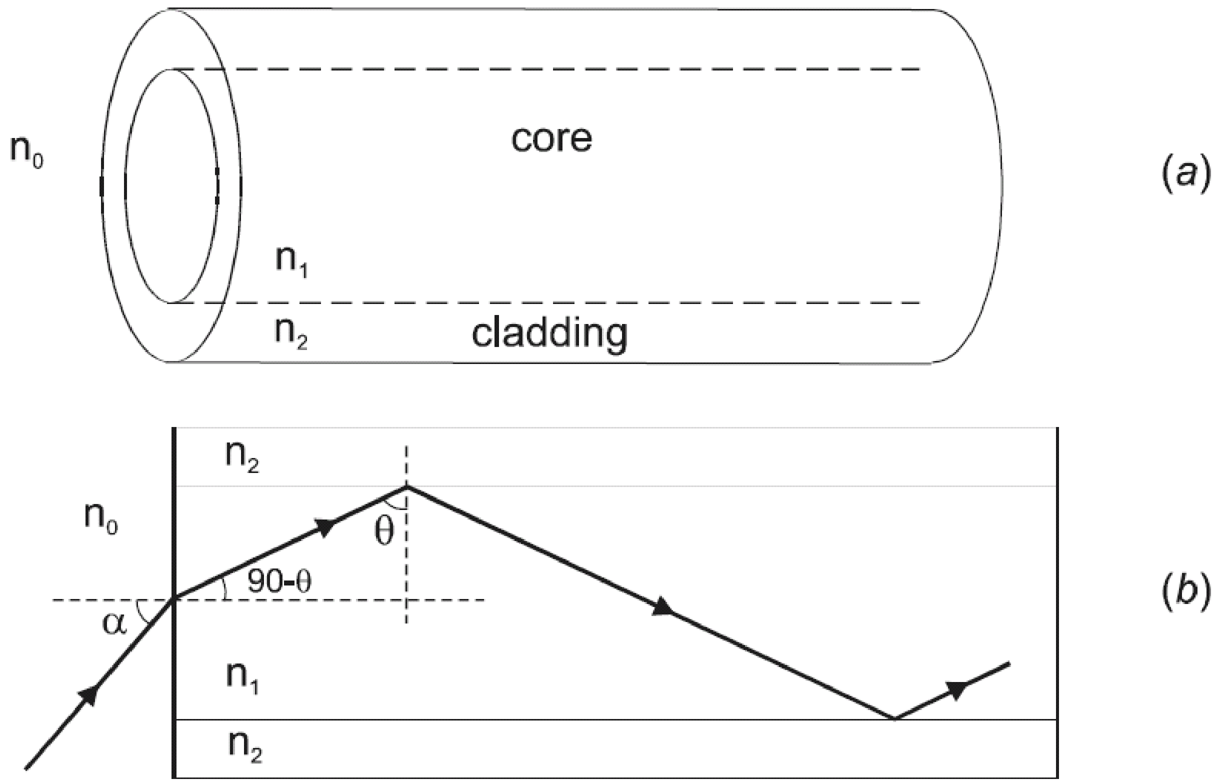



**Figure 4-1 (a)** Perspective view of optical fiber core-cladding structure.
**(b)** Side view of optical fiber, showing path of a light ray that enters the fiber end.

# 4-1 Acceptance Angle and Numerical Aperture

The fiber will accept light into the guided modes only for entrance angles within the range $0 < \boldsymbol{\alpha} < \alpha_{max}$.

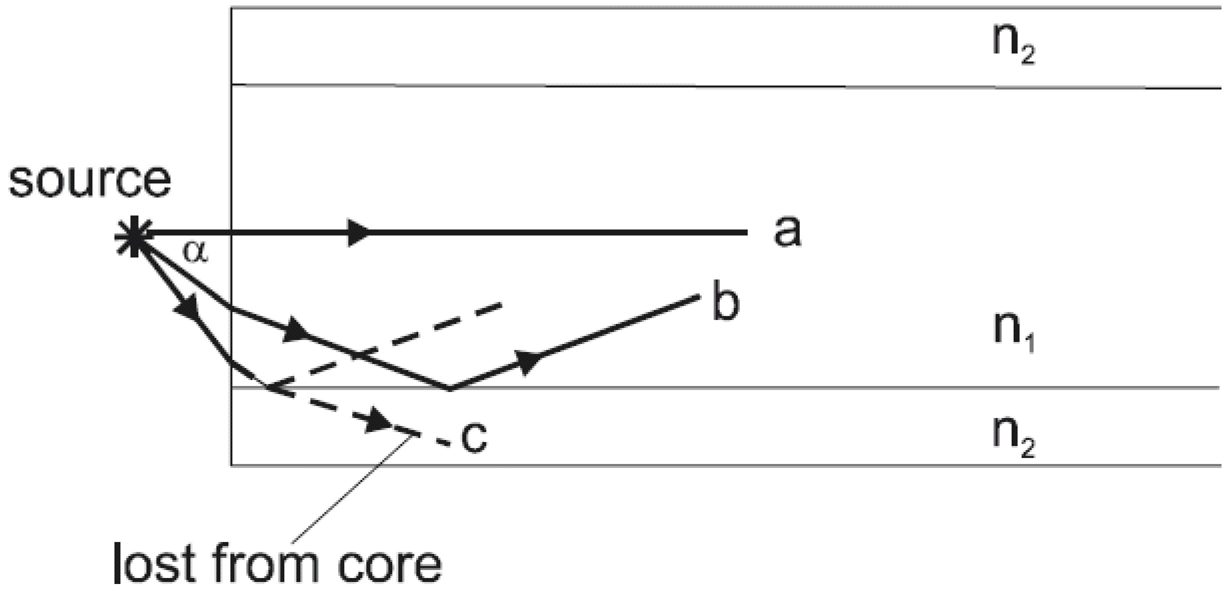



**Figure 4-2** Rays incident over some range of angles are coupled into propagating modes (a and b). Other rays (c) are attenuated by partial transmission at core–cladding boundary.

# 4-1 Acceptance Angle and Numerical Aperture

In **three dimensions** for a cylindrical fiber, this corresponds to an acceptance angle cone of half-angle max.

$$n_0 \sin \alpha_{\text{max}} = n_1 \sin(90 - \theta_c) \qquad \text{(4-1)}$$

$$\sin(90 - \theta_c) = \cos(\theta_c),\ \cos(\theta_c) = \sqrt{1 - \sin^2\theta_c},$$

$$\sin \theta_c = n_2/n_1$$

$$\boxed{n_0 \sin \alpha_{\text{max}} = \text{NA}} \qquad \text{(4-2)}$$

*numerical aperture NA*

$$\boxed{\text{NA} \equiv \sqrt{n_1^2 - n_2^2}} \qquad \text{(4-3)}$$

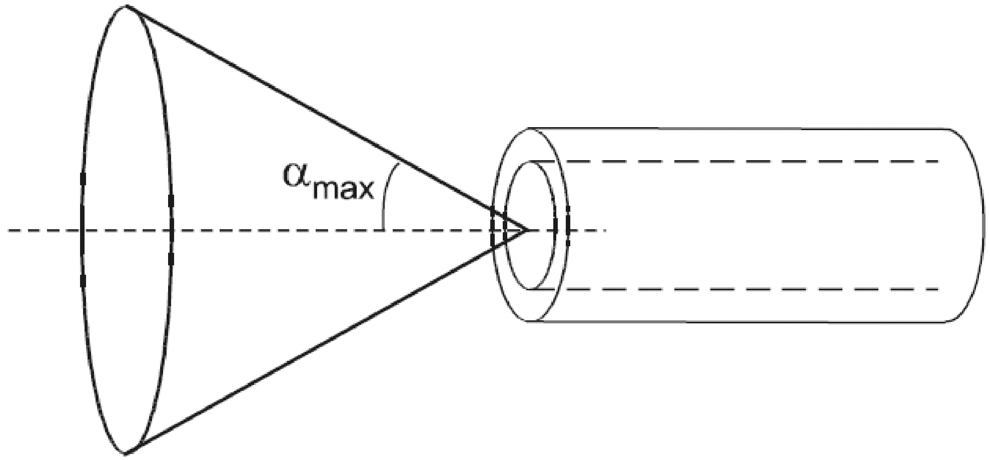



**Figure 4-3** Light enters or exits the fiber within a **cone** of **half-angle** $\alpha_{\text{max}}$.

# 4-1 Acceptance Angle and Numerical Aperture

*multimode fibers* case of wide core diameters,

*single-mode fibers* For small core diameters

fractional index difference $$\boxed{\Delta = (n_1 - n_2)/n_1}$$ (4-3)

A typical value for telecommunications fiber being $\Delta \approx 0.01$.

we can define a single approximate index $n \approx n_1 \approx n_2$.

$$\text{NA} \equiv \sqrt{n_1^2 - n_2^2} \Rightarrow \boxed{\text{NA} \simeq n\sqrt{2\Delta}}$$ (4-5)

Germanium is often added to the core glass in an optical fiber to raise its **refractive index.** Adding 20% Ge by weight gives $\Delta n \approx 0.025$.

# EXAMPLE 4-1

- Determine the numerical aperture and acceptance angle for a multimode fiber with core index 1.5 and fractional index difference 0.01,
  - assuming that light is incident on the fiber from air. Repeat if the fiber is immersed in water (index 1.33).

$$\boxed{\Delta = (n_1 - n_2)/n_1}$$

$$\boxed{\mathrm{NA} \simeq n\sqrt{2\Delta}}$$

$$\boxed{n_0 \sin \alpha_{\max} = \mathrm{NA}}$$

***Solution:*** *for the fiber in either air or water*

$$\mathrm{NA} \simeq (1.5)\sqrt{(2)(0.01)} = 0.21$$

$$\alpha_{\max} = \sin^{-1}(0.21) = 12^\circ \qquad \text{(in air)}$$

$$\alpha_{\max} = (1/1.33)\sin^{-1}(0.21) = 9^\circ \qquad \text{(in water)}$$

# 4-2. Cylindrical Waveguide Modes

The problem of determining the allowed modes in a cylindrical geometry is similar in principle to that of the planar waveguide, but the mathematical treatment is much more complex.

$$E_{\text{mode}} = E_0 g(r, \phi) e^{i(\omega t - \beta z)}$$

$\boxed{d = 2a.}$

The solution for *g(r, Φ) turns out to be in the form of Bessel functions in the radial (r)* direction and sinusoidal functions in the azimuthal (*Φ*) direction.

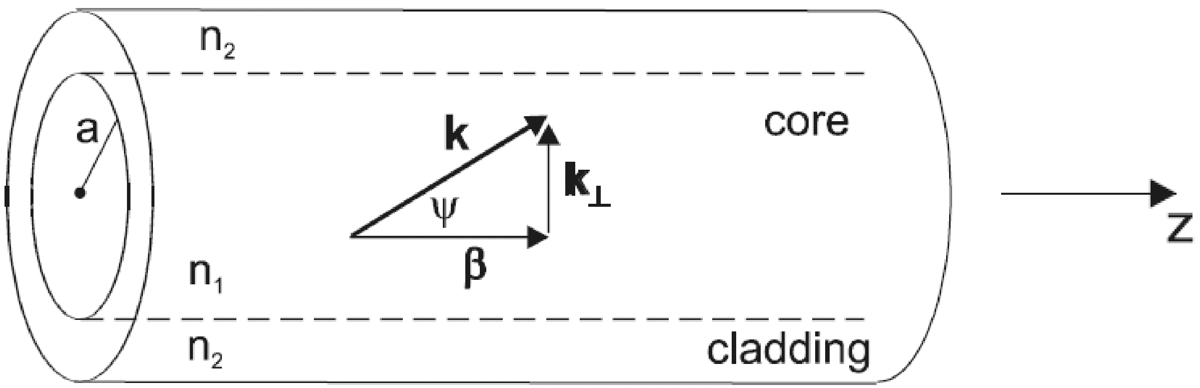



**Figure 4-4** Fiber geometry with core radius *a, core index $n_1$, and cladding index $n_2$*

# Number of Modes (1)

$$\mathbf{k} = \mathbf{k}_{\parallel} + \mathbf{k}_{\perp}$$

$$\mathbf{k}_{\parallel} = \beta \hat{k}$$

$$\mathbf{k}_{\perp} = k_x \hat{i} + k_y \hat{j}$$

$$k_x 2d = m2\pi$$

$$k_y 2d = l2\pi$$

The allowed values of $k_x$ *and* $k_y$

$$k_x = m\frac{\pi}{d}, \qquad k_y = l\frac{\pi}{d}$$

where $\hat{i}$, $\hat{j}$, *and* $\hat{k}$ *are the usual unit vectors.*

where *m and l are integers.*

$$d = 2a.$$

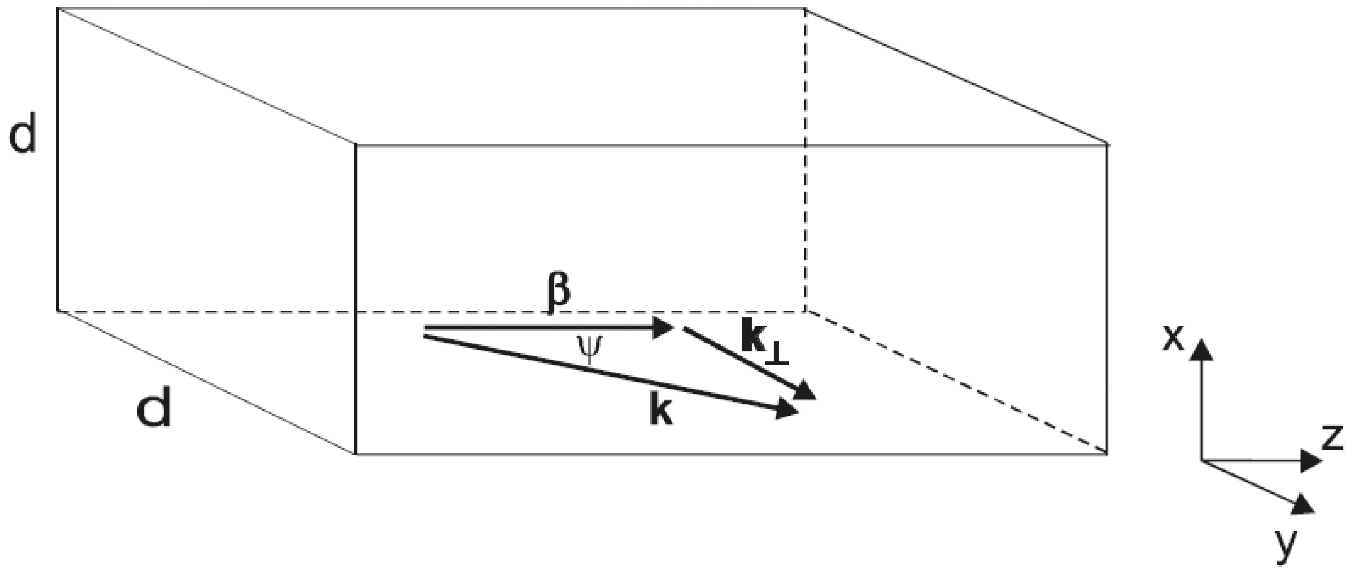



**Figure 4-5** Rectangular approximation for fiber, with width **d = 2a**.

# Number of Modes (2)

- The **waveguide modes** can be represented as points in the two-dimensional k space.

- The upper limits on *m and l are* found by requiring that total internal reflection occur at the **core–cladding boundary**.

Similar to $V_p$ for planar waveguides:

$$V_p \equiv \frac{2\pi d}{\lambda_0}\sqrt{n_1^2 - n_2^2} \qquad \text{[Eq. (3-8)]},$$

$$\boxed{V \equiv \frac{2\pi a}{\lambda_0}\sqrt{n_1^2 - n_2^2}} \qquad (4\text{-}9)$$

$$\boxed{V_p = 2V}$$

$$\boxed{d = 2a.}$$

$$\boxed{k_x = m\frac{\pi}{d}, \qquad k_y = l\frac{\pi}{d}}$$

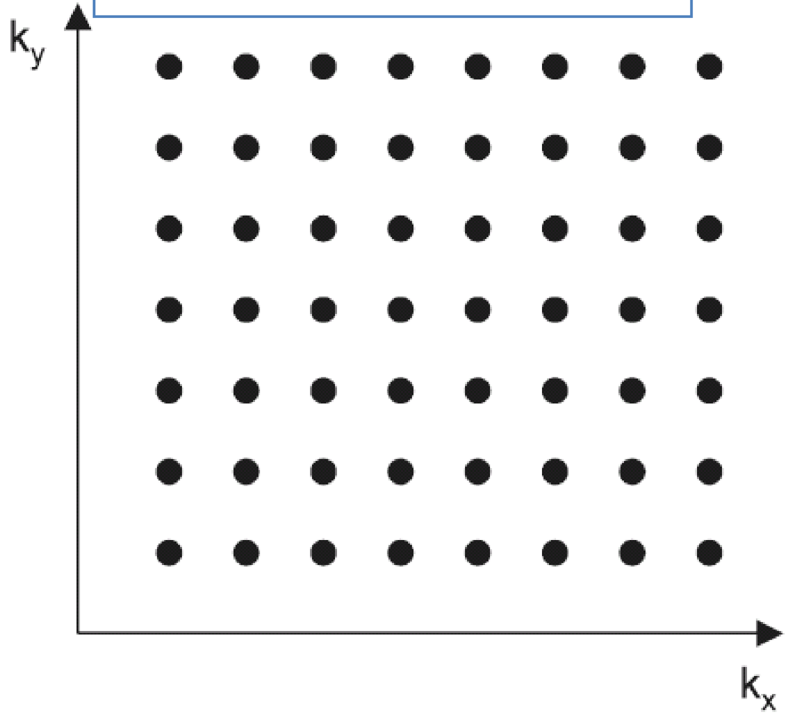



**Figure 4-6** Allowed modes for rectangular waveguides are uniformly spaced in $k_x$–$k_y$ *space*.

# Number of Modes (3)

$$V_p = 2V$$

1-D

$$\boxed{\text{number of modes} = p + 1 = \text{int}\left(\frac{V_p}{\pi}\right) + 1} \quad (3\text{-}10)$$

- The number of guided modes with $k_y = 0$ for large V reduces to ≈ $2V/\pi$.
- Similarly, for modes with $k_x = 0$ there are ≈ $2V/\pi$ guided modes.
- The total number of guided modes for any combination of $k_x$ and $k_y$ is therefore expected to be the product of these two numbers, or ≈ $(2V/\pi)^2$.
- This simple analysis gives the essential feature that: the number of modes is $\sim V^2$ for a 2-D waveguide, rather than $\sim V$ for a 1-D waveguide.

❖ The above calculation overestimates the number of allowed modes, because modes with $k = (k_x^{\,2} + k_y^{\,2})^{1/2} > V/a$ *are not guided.*

$$\#\ \text{modes (rough estimate)} \approx \frac{2}{\pi} V^2 \quad (4\text{-}10)$$

$$\#\ \text{modes (actual value)} \leqslant \frac{1}{2} V^2 \quad (4\text{-}11)$$

# EXAMPLE 4-2

❑ **How many modes** can propagate in a step-index fiber with a 100 μm **diameter core** and $\Delta = 0.03$? Take the core **index of refraction** as 1.5 and the free-space wavelength as 1.00 μm.

***Solution:***

The core radius is a = D/2 = 50 μm, and the V parameter is:

$$V \simeq \frac{2\pi a n\sqrt{2\Delta}}{\lambda_0} = \frac{2\pi\,(50\cdot 10^{-6})(1.5)}{10^{-6}}\sqrt{2(0.03)}$$

$$V \simeq 115$$

$$\#\,\text{modes} \simeq \frac{1}{2}V^2 = 6{,}660$$

# Mode Patterns (1)

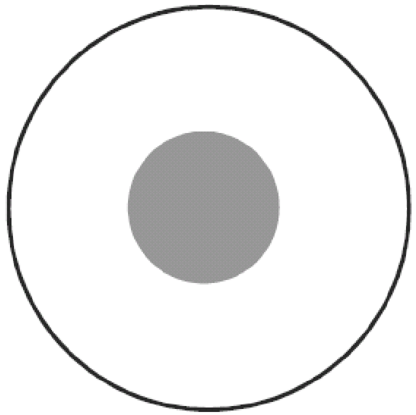
- The mode pattern for a rectangular waveguide would **consist** of standing waves in the $x$ and $y$ directions, with propagating waves in the $z$ direction.

- Larger **mode numbers** $m$ and $l$ mean a more rapid variation in intensity with $x$ and y, giving rise to $m$ maxima in the $x$ direction and $\approx l$ maxima in the $y$ direction.
- The situation is qualitatively the same for fibers with cylindrical symmetry, the difference being the symmetry and shape of the resulting modes.

LP: linearly polarized

**Figure 4-7** Typical mode patterns for the $LP_{lm}$ modes.

(a) $LP_{23}$

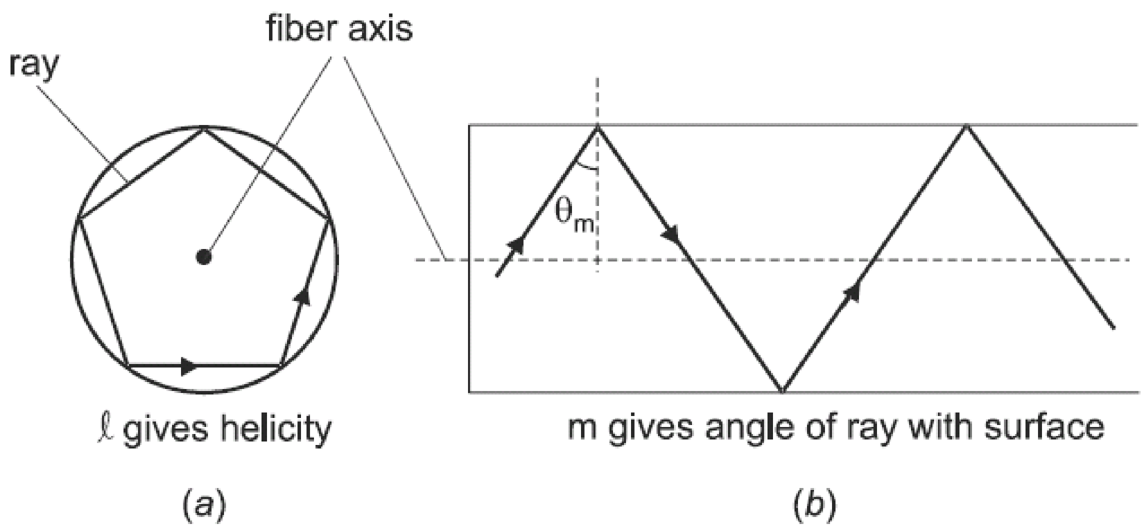


(b) $LP_{01}$ ($HE_{11}$)

# Mode Patterns (2)

- There are *m* **maxima** in the mode intensity along a radial direction, and *2l* **maxima** along the circumference of a circle around the fiber center.
- In the ray picture, m corresponds to rays making different angles with the fiber axis.
- Likewise, the integer l corresponds to the helicity (tightness of the spiral) of the ray as it corkscrews down the fiber.

**Figure 4-8**
(a) End view of fiber, showing spiraling of skewed rays around the fiber axis.
(b) Side view.



## Single-Mode Fibers (1)

- Fro planar waveguides: single mode was allowed when $Vp < \pi$.
- Setting $Vp = 2V$ as before, we might expect the *single-mode condition for a fiber to be* $V \sim \pi/2$.
- **The actual result of a rigorous treatment is close to this**:

$$\boxed{V < 2.405 \qquad \text{(single-mode condition in fiber)}} \qquad (4\text{-}12)$$

where

$$\boxed{V \equiv \frac{2\pi a}{\lambda_0}\sqrt{n_1^2 - n_2^2}}$$

Eq. (4-12) can be written in the form:

$$\boxed{\frac{2\pi a}{\lambda_0}\,\mathrm{NA} < 2.405 \qquad \text{(single-mode condition)}} \qquad (4\text{-}13)$$

# Single-Mode Fibers (2)

$$\boxed{\frac{2\pi a}{\lambda_0}\,\mathrm{NA} < 2.405 \qquad \text{(single-mode condition)}} \qquad (4\text{-}13)$$

1. The core radius, $a$, 2. Numerical aperture NA of the fiber,
3. The optical wavelength $\lambda_0$.

- In principle, either $a$ or NA could be reduced to achieve single-mode operation.
- In practice, NA ~ 0.20 is typical for fibers used in optical communications.

The wavelength at which the fiber just becomes single-mode is termed the *cutoff wavelength* $\lambda_c$, defined by

$$2.405 = \frac{2\pi a}{\lambda_c}\,\mathrm{NA} \Rightarrow \boxed{\lambda_c = \frac{2\pi a\,\mathrm{NA}}{2.405} \qquad \text{(cutoff wavelength)}} \qquad (4\text{-}14)$$

❖ For wavelengths $\lambda > \lambda_c$, the fiber will be single-mode, whereas for $\lambda < \lambda_c$ it will be multimode.

# EXAMPLE 4-3

**a)** How small must the core be if only one mode is to propagate in a fiber with $\Delta = 0.01$?

Take the core index of refraction as 1.5 and the free-space wavelength as 1.00μm.

***Solution:***

For $\Delta << 1$, we use the approximation $NA \approx n\sqrt{(2\Delta)} = 0.212$.

$$\boxed{\frac{2\pi a}{\lambda_0}\,\mathrm{NA} < 2.405 \qquad \text{(single-mode condition)}}$$

$$\Rightarrow \quad a < \frac{(2.405)\lambda}{2\pi \mathrm{NA}} = \frac{(2.405)(1\ \mu\mathrm{m})}{2\pi(0.212)} = 1.8\ \mu\mathrm{m}$$

The core diameter must therefore be less than 3.6 μm.

# EXAMPLE 4-3 (continue)

**b)** A fiber with the **same** $\Delta$ and n has a core diameter of 4.4 μm. For what range of wavelengths will the fiber be single-mode?

***Solution:***

$$\boxed{\lambda_c = \frac{2\pi a \text{ NA}}{2.405} \qquad \text{(cutoff wavelength)}}$$

$$\Rightarrow \quad \lambda_c = \frac{2\pi a \text{NA}}{2.405} = 1.22\ \mu\text{m}$$

This fiber would therefore be single-mode for $\lambda > 1.22\ \mu\text{m}$.

# Mode Chart (1)

- The modes in a planar waveguide can be described by an effective refractive index:

$$n_{\text{eff}} = \frac{\lambda_0}{2\pi}\beta \qquad (4\text{-}15)$$

where $\beta$ is the axial wave vector and $\lambda_0$ is the free-space wavelength of light.

- Similarly, one can determine the effective index for an optical fiber, defined by Eq. (4-15), as a function of the core radius *a*.

- The resulting variation in $n_{\text{eff}}$ for a fiber is shown in Fig. 4-9, plotted versus the dimensionless V parameter.

- Qualitatively, the mode charts for the fiber and planar waveguide are similar.

# Mode Chart (2)

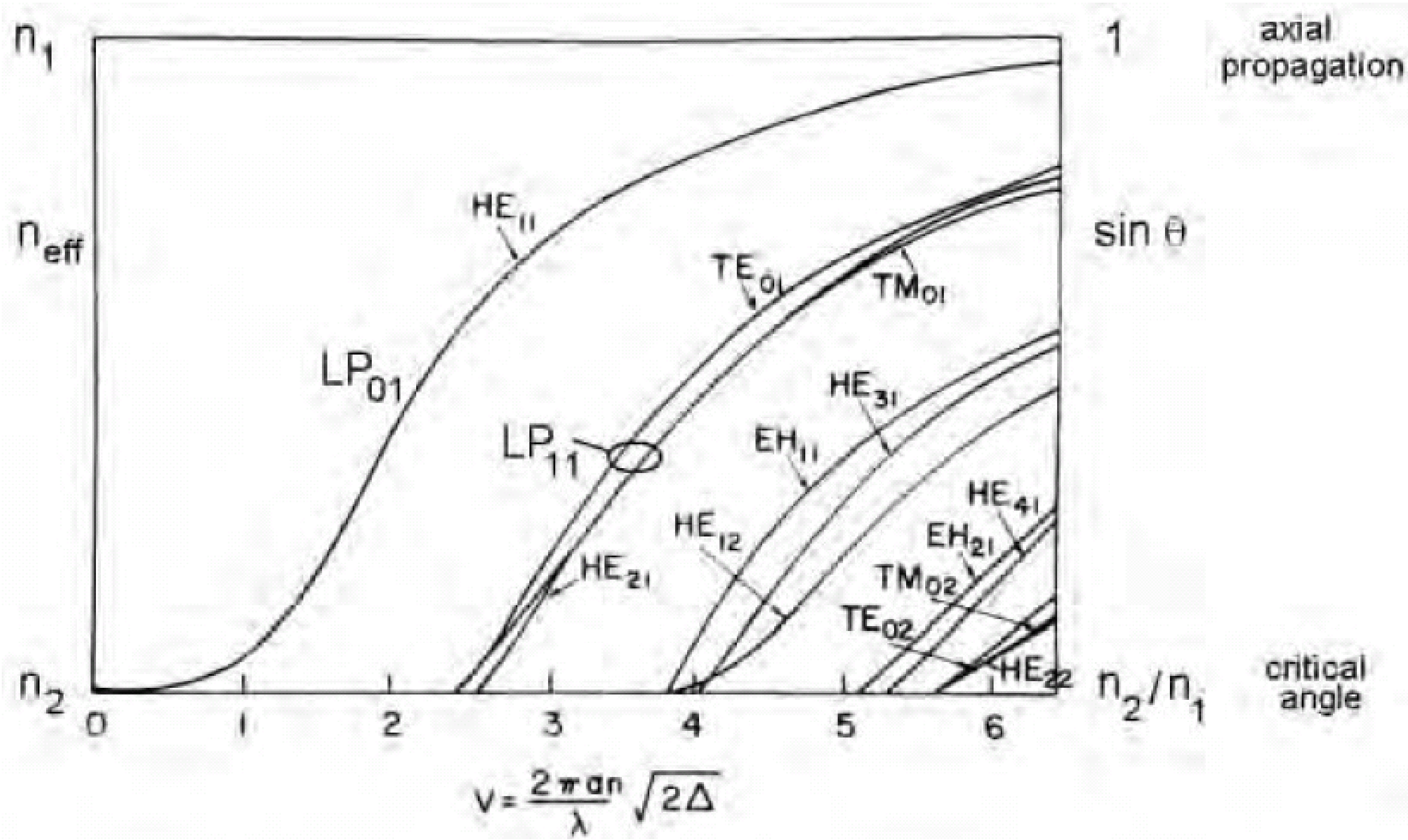



**Figure 4-9** Mode chart for optical fiber

# Mode Chart (3)

1. **In each case**, there is one mode that can propagate for an arbitrary small **waveguide** width.
2. As the waveguide **width** increases, ***other modes*** (which were "cut off" for smaller V) become ***propagating*** as well.
3. $$V < 2.405 \qquad \text{(single-mode condition in fiber)}$$
4. The number of **propagating** modes is found by drawing a vertical line at a particular value of V and counting the number of mode lines that are crossed.
5. ***Some of the modes*** are "**degenerate**," in the sense that they have the same value of $n_{\text{eff}}$ for a given V.
6. ***For example***, for $2.6 < V < 3.8$ there are four mode line crossings, for a **total** of eight modes (including two polarizations for each **spatial mode**).

# Mode Chart (4)

7. The nomenclature for the modes in a circular waveguide depends on the level of approximation.
8. **Most** optical fibers can be considered to be "weakly guiding," with $\Delta \ll 1$.
9. The ***modes*** *in an exact treatment are* ***labeled*** *TE, TM, HE, and EH.*
10. The TE and TM **modes** are analogous to the **transverse** electric and **transverse** magnetic **modes.** **(**$E_z = 0$ (TE) or $B_z = 0$ (TM)).
11. The HE and EH **modes** are hybrid modes in which both $E_z$ and $B_z$ are nonzero.
12. In the optical fiber there are no modes that are **truely** TEM (transverse electric and magnetic), with $E_z = B_z = 0$.
13. However, because of the weak guiding, light propagation in the fiber is paraxial (*close to the fiber axis*), and the axial components $E_z$ and $B_z$ are small.

# Mode Chart (5)

14. The next level of approximation is to neglect these axial fields and consider the modes to be TEM.
15. These are the *linearly polarized modes*, $LP_{lm}$, the patterns for which were shown in Fig. 4-7.
16. The lowest order mode is $LP_{01}$, which corresponds to the $HE_{11}$ hybrid mode.
17. The next-highest mode is $LP_{11}$, which corresponds to the three modes $HE_{21}$, $TE_{01}$, and $TM_{01}$.
18. Similar groupings occur for the higher-order modes. For most applications in which $\Delta << 1$, the LP approximation is adequate.

# Gaussian Mode Approximation (1)

(4-16)

The ***lowest-order mode*** $LP_{01}$ in a single-mode fiber is found to be approximately *Gaussian,* with electric field varying with radial distance *r from the fiber axis as:*

$$E(r) = E_0\, e^{-(r/\omega)^2} \qquad \text{(Gaussian profile)}$$

*w: mode waist size,*
*2w: mode field diameter*

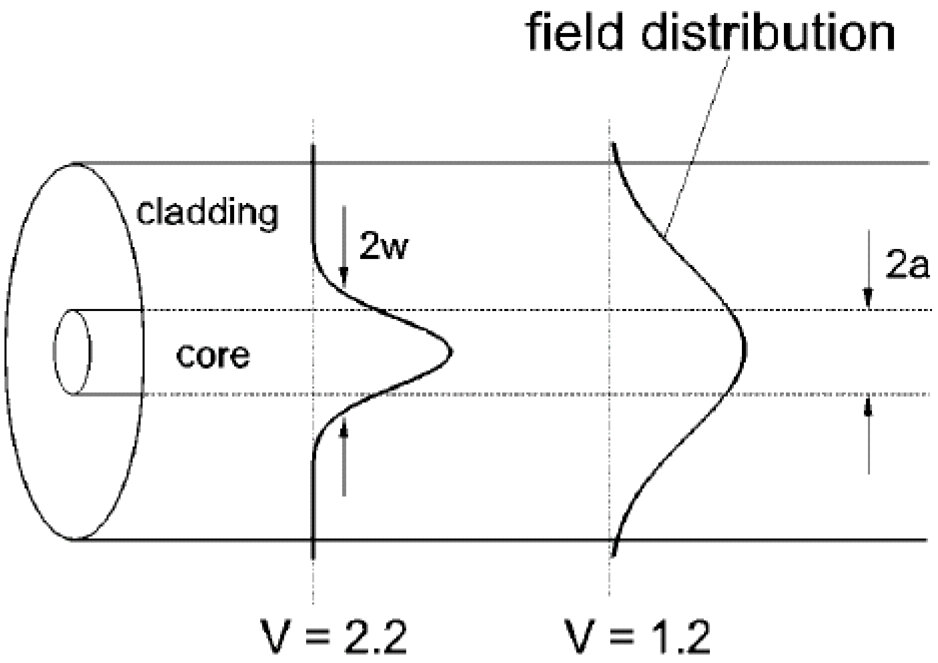



**Figure 4-10** Gaussian modes with *mode field diameter* 2w in fiber of diameter 2a. Shown is a tightly confined mode with V = 2.2, and a loosely **confined mode with V = 1.2.**

# Gaussian Mode Approximation (2)

➢ **Optical intensity** I varies $E^2$,

$$E(r) = E_0\, e^{-(r/\omega)^2} \quad \Rightarrow \quad I(r) = I_0\, e^{-2(r/\omega)^2} \qquad (4\text{-}17)$$

For $1.2 < V < 2.4$, the mode waist size is approximately given by [Ref.]

*a: fiber core radius*

$$w \simeq a\left(0.65 + \frac{1.619}{V^{1.5}} + \frac{2.879}{V^{6}}\right) \qquad (4\text{-}18)$$

❖ As *V* ***decreases,(by*** *decreasing the* ***core radius),*** the mode waist size increases.

➢ As the core is made smaller to confine the light, ***diffraction*** becomes **more important** and **acts** to **resist** that confinement.

# Gaussian Mode Approximation (3)

$$E(r) = E_0\, e^{-(r/\omega)^2}$$

- **Diffraction phenomenon:**

The **steady-state** mode distribution given by Eq. (4-16) is a result of the balance between the ***tendency*** of the *high-index core* to confine the beam and the ***tendency*** of *diffraction* to *spread* the beam out.

- A fiber with very small V has a large ratio of w/a, and the resulting mode is referred to as weakly guided.

- The beam may become so spread out that the **majority** of the **mode's energy** is contained in the cladding region rather than the core.

- This behavior can be useful for **various devices** such as *fiber sensors* and *fiber couplers*, but is detrimental for low loss communications fiber.

- For best confinement of the **mode** the fiber V is often chosen to be not much below the cutoff value of 2.405.

# Gaussian Mode Approximation (4)

- A commonly used *telecommunications fiber* designed for 1550 nm has parameters $2a = 8.3\ \mu m$ and $\Delta = 0.0036$.

$$\boxed{\mathrm{NA} \simeq n\sqrt{2\Delta}} \quad (4\text{-}5) \quad \Longrightarrow \quad \mathrm{NA} \approx 0.13$$

$$\boxed{V \equiv \frac{2\pi a}{\lambda_0}\sqrt{n_1^2 - n_2^2}} \quad (4\text{-}9) \quad \Longrightarrow \quad V \approx 2.19$$

$$\boxed{w \simeq a\left(0.65 + \frac{1.619}{V^{1.5}} + \frac{2.879}{V^6}\right)} \quad (4\text{-}18) \quad \Longrightarrow \quad 2w = 9.8\ \mu m$$

- This is only 18% *higher than the actual core diameter,* which means that the mode is well confined by the core.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