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Low-cost finite element method-based reliability analysis using adjusted control variate technique

Mohsen Rashki [a,*], Alireza Ghavidel [b], Hamed Ghohani Arab [b], Seyed Roohollah Mousavi [b]

[a] Department of Architecture Engineering, University of Sistan and Baluchestan, Zahedan, Iran
[b] Civil Engineering Department, University of Sistan and Baluchestan, Zahedan, Iran

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ABSTRACT

Reliability analysis is used to evaluate the safety of engineering structures subject to uncertainties. Finite element method (FEM) is a popular engineering tool used to evaluate the reliability of complex engineering structures. In general, FEM–based reliability analysis of engineering structures is influenced by the mesh density of the model and the accuracy of the results requires the use of a very fine mesh density in the analysis. However, it is often impractical for reliability analysis complex structures, especially those with low failure probabilities. Hence, a new method is proposed to address this issue, which provides an accurate estimate of the failure probability at low computational cost. In this method, the control variate technique is used in conjunction with the FEM-based reliability analysis, where the failure probability integral is broken down into two separate integral terms. The first term provides a low-cost estimate of the failure probability using a model with coarse mesh density, whereas the second term regulates the failure probability based on fewer finite element analyses with fine mesh density. The adjusted correction factors are also presented in this paper in order to improve the efficiency of the proposed approach. The proposed approach is used to estimate the reliability index of four engineering structures and the results show that the method is efficient and practical for FEM-based reliability analysis of engineering structures.

1. Introduction

A fundamental problem in structural reliability theory is to compute the failure probability (P_f), which is a multifold probability integral defined as:

$$P_f = Prob[g(x) \leqslant 0] = \int_{g(x)\leqslant 0} f(x)dx, \tag{1}$$

where x is a vector of random variables representing uncertain structural quantities. The functions $g(x)$ and $f(x)$ denote the limit state function and the joint probability density function (PDF) of x, respectively.

In most engineering applications, the multifold probability integral given by Eq. (1) is difficult to compute because it involves multi-dimensional integration, where the dimension equals to the number of basic random variables.

Various analytical and simulation methods have been developed over the years to solve the integral above. First order reliability methods (FORMs) are typically used to estimate the failure probability without incurring long computational processing time [1–4]. However, the main disadvantage of these methods is that they often do not yield accurate results for cases involving non-normal distributions, limit state functions that are highly nonlinear, multiple basic variables, and complex failure surfaces [5]. For this reason, a number of simulation methods have been developed to compute the failure probability with high accuracy [5–7]. One of these methods is Monte Carlo simulation (MCS), which involves generating random samples based on the mean value of the variables [6].

For small failure probabilities, the MCS method is a rather time-consuming approach due to the large number of samples required [6–8]. This disadvantage may be eliminated by using an instrumental PDF, $h(x)$, to generate more samples within the failure region:

$$P_f = \int_{g(x)\leqslant 0} \left(\frac{f_x(x)}{h_x(x)}\right) h_x(x)dx, \tag{2}$$

* Corresponding author at: Department of Architecture, University of Sistan and Baluchestan, P.O. Box 9816745563-161, Zahedan, Iran.

E-mail addresses: Mrashki@eng.usb.ac.ir (M. Rashki), a.r.ghavidel@pgs.usb.ac.ir (A. Ghavidel), Ghohani@eng.usb.ac.ir (H. Ghohani Arab), s.r.mousavi@eng.usb.ac.ir (S.R. Mousavi).

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This method is known as importance sampling (IS). The importance sampling estimator is given by [9,10]:

$$P_f = \frac{1}{N}\sum_{i=1}^{N} I[g(x_i)]\frac{f(x_i)}{h(x_i)}, \tag{3}$$

The weighted average simulation method (WASM) is also an efficient simulation method to compute the failure probability and determine the most probable point (MPP) [2,6–8]. In this method, random numbers are first generated based on the initial assumption of the failure probability. Next, a weight index is assigned to the generated samples based on their competency and the failure probability is estimated using the following equation:

$$P_f = \frac{\sum_{i=1}^{N} I[g(x_i)](\prod_{j=1}^{s} f_j(i))}{\sum_{i=1}^{N}(\prod_{j=1}^{s} f_j(i))}, \tag{4}$$

In addition, other methods such as line sampling (LS), subset simulation (SS), metamodel line sampling, and unbiased metamodel method have been developed to overcome the limitations of MCS for various engineering problems [11–14].

However, these methods may not be feasible in practice when the performance function needs to be solved using a time-consuming approach such as finite element method (FEM).

In deterministic analyses, mesh convergence analysis [15,16] and grid convergence index (GCI) [17–20] are used to select the suitable finite element (FE) model and the simulation results are compared with those obtained from analytical functions or experiments [21–24]. Once mesh convergence is achieved, the differences in the FE results obtained from different mesh densities will be small and these small errors are considered as acceptable. Several researchers have assessed the effects of mesh density in deterministic analyses [25,26]. For example, Waide et al. [27] investigated the load transfer characteristics of two types of cemented hip replacements with fibrous tissue layer using FEM and they compared the results with those obtained from experiments. The results showed that the maximum difference between the FE and experimental results was 15%. In addition, one study on spinal segments showed that the difference in the FE results was less than 5% once mesh convergence was attained, which the researchers perceived as adequate [28].

However, there are very few studies focused on the selection of a suitable FE model for probabilistic reliability analysis [29].

This study shows that the errors considered as acceptable in deterministic analysis (errors arising from inadequate mesh density) have a significant effect on the evaluating the safety of engineering structures and very fine mesh densities are required to estimate the failure probability with reasonable accuracy. Owing to the fact that it is impractical and time-consuming to use models with very fine mesh densities in reliability analysis, a new FEM-based reliability analysis method is proposed in this study to compute the reliability index in a simple, efficient manner with a high degree of accuracy and low computational cost.

2. Development of the adjusted control variate technique (ACVAT) for FEM-based reliability analysis of engineering structures

When the performance evaluation of an engineering structure requires the use of FEM, Eq. (1) which is used to compute the failure probability can be written as:

$$P_f = \int_{g\leqslant 0} f(x)dx \cong \int_{G^{FEA}\leqslant 0} f(x)dx, \tag{5}$$

where $G^{FEA} \leqslant 0$ is the failure region. The performance of the structure in this domain is evaluated by FEM.

Even though it is possible to determine errors due to inadequacy of mesh density in deterministic analysis, it is challenging to determine errors inherent in FEM-based reliability analysis. Thus, the control variate technique (CVT) is adopted in this study to tackle this issue.

Suppose that the objective is to estimate the following failure probability integral:

$$\mathbb{E}(p) = \int p(x)f(x)dx, \tag{6}$$

where $p(x)$ is the function of interest and $f(x)$ is the PDF of the input x. When the function $p(x)$ is not known or complex, estimation of the failure probability integral becomes difficult. Hence, in the CVT, it is assumed that there is another function $g(x)$, which is correlated with $p(x)$ with a known mean. Hence, Eq. (6) can be approximated as [30]:

$$\mathbb{E}(p) = \int g(x)f(x)dx + \int (p(x) - g(x))f(x)dx. \tag{7}$$

In this formulation, $g(x)$ is known as the control variate for $p(x)$. Since the mean of the first term is known (or estimating its expectation is easier than $p(x)$), the method transfers the difficulty of the estimation to the second term. Indeed, $p(x)$ effects on the total estimation are reduced.

For FEM-based reliability problems, Eq. (1) can be written as:

$$P_f = \int_{g\leqslant 0} f(x)dx \cong \int_{-\infty}^{+\infty} p\left(G_{fine}^{FEA}\right)f(x)dx, \tag{8}$$

where

$$p\left(G_{fine}^{FEA}\right) = \begin{cases} 1, & G_{fine}^{FEA} \leqslant 0 \\ 0, & G_{fine}^{FEA} > 0 \end{cases} \tag{9}$$

In this equation, G_{fine}^{FEA} represents the performance function evaluated by the FE model with a very fine mesh density. Solving reliability problems with this specification is impractical for complex engineering problems. Hence, in the proposed approach (i.e., ACVAT), the results obtained from the FE model with coarse mesh density are used as the control variates of the FE model with fine mesh density. Hence, the failure probability integral given by Eq. (7) can be rewritten as:

$$\begin{gathered} P_f = \int g\left(G_{coarse}^{FEA}\right)f(x)dx + \int \left(p\left(G_{fine}^{FEA}\right) - g\left(G_{coarse}^{FEA}\right)\right)f(x)dx, \\ p\left(G_{coarse}^{FEA}\right) = \begin{cases} 1, & G_{coarse}^{FEA} \leqslant 0 \\ 0, & G_{coarse}^{FEA} > 0 \end{cases} \end{gathered} \tag{10}$$

where G_{Coarse}^{FEA} represents the performance function evaluated by the FE model with coarse mesh density. It shall be noted that $g\left(G_{Coarse}^{FEA}\right)$ is the control variate of $p\left(G_{fine}^{FEA}\right)$. Hence, Eq. (10) is rewritten as:

$$P_f = \int g(G_{coarse}^{FEA})f(x)dx + \frac{\sum\left(p(G_{fine}^{FEA}) - g(G_{coarse}^{FEA})\right)}{N}, \tag{11}$$

where the sampling for estimation of the second term is performed based on $f(x)$ and N represents the sample size. The first term is the failure probability of the given problem, which is solved using the FE model with coarse mesh density, i.e., $\int g\left(G_{Coarse}^{FEA}\right)f(x)dx = \mathbb{E}(g\left(G_{Coarse}^{FEA}\right)) = P_f^{Coarse}$. Estimating the first term requires lower computational cost compared with estimating an accurate P_f. Accordingly, Eq. (11) may be written in the following form:

$$P_f = P_f^{coarse} + \frac{\sum \left(p\left(G_{fine}^{FEA}\right) - g\left(G_{coarse}^{FEA}\right)\right)}{N}, \tag{12}$$

with the following variance:

$$\begin{aligned} Var(P_f) &= Var\left(P_f^{Coarse}\right) + Var\left(\frac{\sum_{i=1}^{n}\left(p\left(G_{fine}^{FEA}\right) - g\left(G_{Coarse}^{FEA}\right)\right)}{N}\right) \\ &\approx \frac{1}{N^2}\sum_{i=1}^{n} Var\left(p\left(G_{fine}^{FEA}\right) - g\left(G_{Coarse}^{FEA}\right)\right) = \frac{1}{N} Var\left(p\left(G_{fine}^{FEA}\right) - g\left(G_{Coarse}^{FEA}\right)\right) \\ &= \frac{1}{N}\left[Var\left(p(G_{fine}^{FEA})\right) + Var\left(g(G_{Coarse}^{FEA})\right) - 2\cdot Cov\left(p\left(G_{fine}^{FEA}\right), g\left(G_{Coarse}^{FEA}\right)\right)\right] \\ &= Var\left(\mathbb{E}\left(G_{fine}^{FEA}\right)\right) + \frac{1}{N}\left[Var\left(g\left(G_{Coarse}^{FEA}\right)\right) - 2\cdot Cov\left(p(G_{fine}^{FEA}), g(G_{Coarse}^{FEA})\right)\right] \end{aligned} \tag{13}$$

To ensure efficiency and accuracy of the ACVAT, the coarse mesh density model needs to satisfy the following criterion: $\mathrm{Var}\left(g\left(G_{Coarse}^{FEA}\right)\right) < 2\cdot Cov\left(p\left(G_{fine}^{FEA}\right), g\left(G_{Coarse}^{FEA}\right)\right)$.

It is relatively easy to determine the FE models with coarse and fine mesh densities using the proposed specification for linear and mildly nonlinear problems. However, it shall be noted that it will be rather arduous to determine the FE models with coarse and fine mesh densities for highly nonlinear FE problems due to the fact that in some cases, FE model with coarse mesh density is unable to fully capture the relevant physics.

In order to estimate P_f in a more efficient manner, the results obtained from the first term can be used to estimate the second term, which is made possible by IS. For this purpose, one can use the MPP and regions with high failure probabilities obtained from the first-term approximation. Let $q(x)$ be the new PDF for IS, where the MPP of the first term may be used as the mean. Hence, Eq. (10) can be written as:

$$P_f = P_f^{coarse} + \int \left(\frac{p(G_{fine}^{FEA})f(x)}{q(x)} - \frac{g(G_{coarse}^{FEA})f(x)}{q(x)}\right) q(x)dx, \tag{14}$$

where the first term is an approximation of the failure probability using the FE model with coarse mesh density and the second term is used to evaluate the sampling results based on $q(x)$. The second term shows the differences between the coarse and fine mesh densities in the critical domains. In practice, the second term regulates the errors resulting from inadequacy of mesh density in estimating the failure probability given by the first term.

It is possible to improve the efficiency of the estimation by applying linear regression on the ACVAT. Without a loss of generality, Eq. (7) may be written as follows:

$$\begin{aligned} \mathbb{E}(p) &= \frac{1}{N}\sum_{k=1}^{N} p(x) - \alpha\cdot\left(\frac{1}{N}\sum_{k=1}^{N} g(x) - \int g(x)f(x)dx\right) \\ &= \alpha\cdot\mathbb{E}(g(x)) + \left(\frac{1}{N}\sum_{k=1}^{N}(p(x) - \alpha g(x))\right) \end{aligned} \tag{15}$$

The equation above is rewritten in the following generic form for FEM-based reliability analysis:

$$\begin{aligned} P_f &= \alpha\cdot P_f^{coarse} + \int\left(\frac{p\left(G_{fine}^{FEA}\right)f(x)}{q(x)} - \alpha\cdot\frac{g\left(G_{coarse}^{FEA}\right)f(x)}{q(x)}\right)q(x)dx \\ &\cong \alpha\cdot P_f^{coarse} + \left(\frac{1}{N}\sum_{k=1}^{N}\left(p\left(G_{fine}^{FEA}\right)(x) - \alpha\cdot g\left(G_{coarse}^{FEA}\right)(x)\right)\right) \end{aligned} \tag{16}$$

where α is a correction factor, which is a constant. The sampling is performed by $q(x)$ around the MPP, which is obtained from the coarse mesh FE analysis. If α is defined appropriately, Eq. (16) will have a lower variance than Eq. (14). There are two ways to estimate the values of α [30–33], which are presented as follows:

2.1. Correction factor #1 (α_1)

In this approach, the optimum value of α is determined by minimizing the variance of the estimation (Eq. (15)) with respect to α [32]:

$$\begin{aligned} \frac{\partial Var(\mathbb{E}(p))}{\partial\alpha} &= 2\alpha\cdot Var(g(x)) + 2Cov(g(x), p(x)) = 0, \Rightarrow \alpha \\ &= -\frac{Cov(g(x), p(x))}{Var(g(x))}. \end{aligned} \tag{17}$$

The adjusted form of α_1 for FEM-based reliability analysis is given by:

$$\alpha_1 = -\frac{Cov(p(G_{fine}^{FEA}), g(G_{coarse}^{FEA}))}{Var(g(G_{coarse}^{FEA}))}. \tag{18}$$

2.2. Correction factor #2 (α_2)

This correction factor was introduced in [33]. The adjusted form of α_2 used in this study is given by:

$$\alpha_2 = -\frac{Cov(g(G_{coarse}^{FEA}) - g(G_{coarse}^{FEA}), p(G_{fine}^{FEA}))}{Var(g(G_{coarse}^{FEA}) - g(G_{coarse}^{FEA})) + b_g^2}, \tag{19}$$

where

$$b_g = \mathbb{E}\left[g\left(G_{coarse}^{FEA}\right) - \hat{g}\left(G_{coarse}^{FEA}\right)\right]. \tag{20}$$

Both of these correction factors improve the efficiency of the simulations and yields results with low variances, as delineated in the references. Hence, it is desirable to apply these correction factors to the ACVAT developed in this study. However, it shall be highlighted that in estimating the second integral term (Eq. (14)) and evaluating the performance (reliability) of the engineering structure, these correction factors may lead to problems in attaining a converged solution when the results of the coarse and fine mesh FE analyses are equal. For this reason, an alternative approach is used to determine the correction factor. The idea here is to remove the second term of Eq. (14) as follows:

$$\begin{aligned} &\int\left(\frac{p(G_{fine}^{FEA})f(x)}{q(x)} - \alpha\cdot\frac{g(G_{coarse}^{FEA})f(x)}{q(x)}\right)q(x)dx = 0, \\ &\int\frac{p(G_{fine}^{FEA})f(x)}{q(x)}q(x) - \alpha\cdot\int\frac{g(G_{coarse}^{FEA})f(x)}{q(x)}q(x) = 0, \\ &\alpha = \frac{\int\frac{p(G_{fine}^{FEA})f(x)}{q(x)}q(x)}{\int\frac{g(G_{coarse}^{FEA})f(x)}{q(x)}q(x)} = \frac{\sum\frac{p(G_{fine}^{FEA})f(x)}{q(x)}}{\sum\frac{g(G_{coarse}^{FEA})f(x)}{q(x)}}. \end{aligned} \tag{21}$$

This correction factor, named α_3, removes the second term of Eq. (14) and calibrates the initial failure probability estimated by the FE model with coarse mesh density:

$$\begin{aligned} P_f &= \alpha_3\cdot P_f^{coarse}, \\ \alpha_3 &= \frac{\sum\frac{p(G_{fine}^{FEA})f(x)}{q(x)}}{\sum\frac{g(G_{coarse}^{FEA})f(x)}{q(x)}}. \end{aligned} \tag{22}$$

The steps involved to solve FEM-based reliability problems using the proposed ACVAT are presented as follows:

Step 1: Perform mesh convergence analysis based on a deterministic point of view.
Step 2: Evaluate the computational cost and accuracy of each FE model. Select a FE model with a reasonably coarse mesh density to determine the failure probability of the engineering structure and select a FE model with fine mesh density (error: ~1%) to minimize errors due to mesh density inadequacy.
Step 3: Estimate the initial failure probability (P_f^{Coarse}), which is the first term of Eq. (14), using the FE model with coarse mesh density and determine the MPP. In simulation methods, the MPP can be simply be determined as the sample with the maximum PDF in the failure region [9].
Step 4: Determine $q(x)$ for sampling based on IS. Use the MPP as the mean of $q(x)$ and generate a few samples.
Step 5: Perform FE analysis with coarse and fine mesh densities for the generated samples in order to determine the correction factors (α_1, α_2, and α_3) and the second term of Eq. (14).
Step 6: Estimate the failure probability of the engineering structure using Eq. (14).
Step 7: Check the convergence of the solution against the desired accuracy using the following formula: $\frac{P_f^{current\ step} - P_f^{previous\ step}}{P_f^{current\ step}}$.
If the solution has converged, stop the simulation. Otherwise, generate a new sample based on $q(x)$ and repeat Steps 5–7.

3. Illustrative examples

3.1. Plate on elastic foundation

In order to demonstrate the applicability of the ACVAT, the reliability of a plate (length × width × thickness: 300 in. × 300 in. × 1 in.) on an elastic foundation is evaluated as the first example. A point load (50 kips) is applied at the center of the plate, as shown in Fig. 1. The performance function of the problem is defined as follows:

$$G = \Delta_{allowable} - \Delta_{Plate}, \tag{23}$$

where $\Delta_{allowable}$ is the maximum allowable deflection and Δ_{Plate} is the maximum deflection of the plate. The analytical function used to compute the exact value of the maximum deflection (Δ_{Exact}) for a plate on elastic foundation is given by [34]:

$$\Delta_{Exact} = \frac{P\lambda^2}{8K}, \tag{24}$$

where

$$\lambda^4 = \frac{K}{D}, \quad D = \frac{Et^3}{12(1-\nu^2)} \tag{25}$$

where D is the flexural rigidity of the plate, K is the subgrade modulus, t is the plate thickness, E is the modulus of elasticity, and ν is the Poisson's ratio. In this study, the maximum allowable deflection is assumed to be $\Delta_{allowable} = 0.25$ in. an explicit performance function of the following form:

$$G = 0.25 - \frac{P}{8}\sqrt{\frac{12(1-\nu^2)}{KEt^3}}. \tag{26}$$

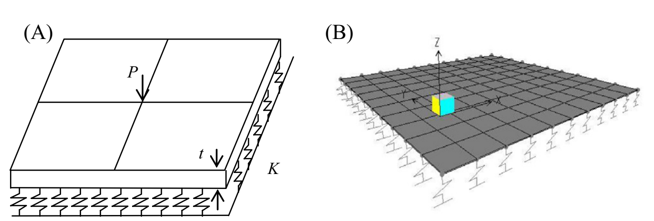

Fig. 1. Plate on elastic foundation with applied point load: (A) Basic model; (B) FE model.

The descriptive statistics of the basic random variables for the plate on elastic foundation are presented in Table 1. In order to verify the accuracy of the proposed approach, MCS is used to compute the reliability of the plate on elastic foundation using the proposed explicit performance function and the reliability index is found to be $\beta = 2.98$.

In order to investigate the effect of mesh density on the FEM-based reliability results, the plate is modeled with different mesh densities (50 × 50, 70 × 70, 90 × 90, and 100 × 100 elements). The performance function of the problem is given by:

$$G = 0.25 - \Delta_{FEM}, \tag{27}$$

where Δ_{FEM} represents the maximum deflection of the plate computed by the FE model. The reliability problem is solved for each FE model independently and the results are presented in Table 2. In this table, $\varepsilon_{Model} = \left|\frac{\Delta_{Exact} - \Delta_{Plate}}{\Delta_{Exact}}\right| \times 100$ represents the error of the FE model as the mean value of the random variables and ε_{PF} represents the error of the FE model in evaluating the reliability of the structure. The scaled central processing unit (CPU) time is also presented for each FE model.

It can be seen from Table 2 that mesh density has a significant effect on the reliability index estimations and a very fine mesh density is required to minimize the error of FEM-based reliability analysis. However, it is undesirable to use a FE model with very fine mesh density for reliability analysis because this will increase the computational processing time by up to 400%.

Hence, the ACVAT is developed in this study to address this issue. Here, the FE model with coarse mesh density (50 × 50 elements) is used as the control variate for the FE model with fine mesh density (100 × 100 elements). The failure probability is approximated by using Eq. (14) and applying the correction factors α_1, α_2 and α_3. A total of 5000 analyses are performed using the FE model with coarse mesh density to compute the initial failure probability (P_f^{Coarse}) whereas 70 analyses are performed using the FE model with fine mesh density to estimate the second term of Eq. (14). With the ACVAT, the initial reliability index obtained using FE model with coarse mesh density is found to be $\beta^{Coarse} = 4.12$. By carrying out fewer analyses using the FE model with fine mesh density, the value of the failure probability is refined and the initial reliability index ($\beta^{Coarse} = 4.12$) becomes $\beta = 3.06$. The results are summarized in Table 3. It can be seen that the accuracy of the solution obtained from the ACVAT shows good agreement with that obtained from 5000 analyses using the FE model with fine mesh density while the computational processing time is reduced by a factor of ~0.25. In addition, 100 independent reliability analysis are performed in order to examine the variance of the ACVAT and the result is also shown in Table 3. It is found that the variance in estimation of the reliability index using the proposed approach is very small.

Table 1
Descriptive statistics of the basic random variables for the plate on elastic foundation.

Variable	Mean	Distribution	C.O.V*
K (kip/ft^3)	800	Normal	0.1 [35]
P (kips)	50	Normal	0.1 [36]
E (kip/in^2)	29000	Normal	0.076 [37]

* C.O.V: Coefficient of variation.

Table 2
Effect of mesh density on the reliability index estimations for the plate on elastic foundation. Note that the results are presented at mean values.

FE model	ε_{Model}	β	ε_{PF}	Scaled CPU time
50 × 50	11.71%	4.12	98.68%	1
70 × 70	5.47%	3.48	82.46%	2.09
90 × 90	2.15%	3.17	46.69%	3.29
100 × 100	1.37%	3.05	18.69%	4.05
Analytical function	0.00	2.98	0.00	–

Table 3
FEM-based reliability results for the plate on elastic foundation.

Parameters	Reliability results				
	5000 coarse mesh FE analyses	5000 fine mesh FE analyses	ACVAT: Coarse mesh results + 70 fine mesh FE analyses		
			Adjusted by α_1	Adjusted by α_2	Adjusted by α_3
$\bar{\beta}$	4.12	3.058	3.067	3.067	3.060
Variance			0.0085	0.0078	0.0170

$\bar{\beta}$: Mean of reliability index.

The effects of IS parameters on the accuracy of the reliability index estimations are also investigated by varying the variance of instrumental sampling function (VIS). For each VIS, the 95% confidence interval is computed from 100 independent reliability analyses using the ACVAT and the results are shown in Fig. 2. It can be observed that the reliability index estimation is quite sensitive to the VIS for the ACVAT adjusted by α_2, unlike the estimations obtained for the ACVAT adjusted by other correction factors. Moreover, the variance in the estimations is higher for the ACVAT adjusted by α_3 compared with those obtained for the ACVAT adjusted by α_1 and α_2. In addition, the accuracy of the reliability index estimation for the ACVAT adjusted by α_3 is not sensitive to the VIS.

In addition, it is found that from the coarse mesh FE analyses that there are variations in the location of the design point and therefore, the effect of this error on the reliability index estimations is investigated in this study. Thus, for this analysis, the location of the design point is multiplied with a number within the interval [0.95, 1.05] and the resulting point is used as the mean in the instrumental sampling PDF. The results are shown in Fig. 3. It can be seen that the proposed correction factors within this interval do not lead to fluctuations in the reliability index estimations.

Fig. 4 shows the effect of sampling size on the reliability indices estimated using Eq. (14). It can be seen that the ACVAT yields good results with a smaller number of samples (about 50 samples). This is certainly advantageous for FEM-based reliability analysis of engineering structures, which can be rather time-consuming and costly.

3.2. Single edge notch test specimen

The reliability of a rectangular plate (dimensions: 60 mm × 30 mm) with a single edge notch (length: 15 mm) is also investigated to demonstrate the applicability of the ACVAT, as shown in Fig. 5. The analytical function used to determine the stress intensity factor is given by [38]:

$$K_I = t\sqrt{\pi a} F\left(\frac{a}{B}\right), \tag{28}$$

for which the numerical value for F is given as follows:

$$F\left(\frac{a}{B}\right) = \sqrt{\frac{2B}{\pi a}\tan\left(\frac{\pi a}{2B}\right)} \cdot \frac{0.752 + 2.02\left(\frac{a}{B}\right) + 0.37\left(1 - \sin\left(\frac{\pi a}{2B}\right)\right)^3}{\cos\left(\frac{\pi a}{2B}\right)}. \tag{29}$$

The single edge notch test specimen is modeled using FEM, as shown in Fig. 6. The simulations are carried out for different mesh densities in order to predict the stress intensity factors and the results are compared with those determined from the analytical function (Eq. (28)), as shown in Table 4. It can be seen that the

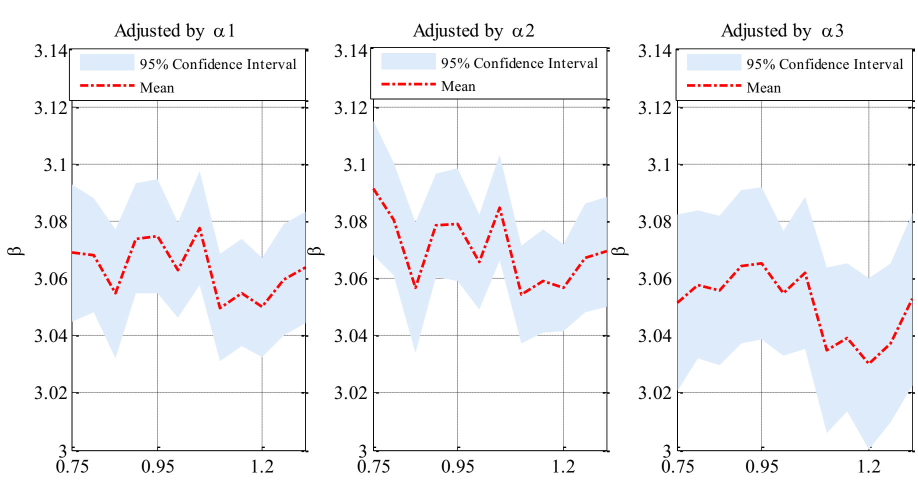

Fig. 2. Sensitivity of the reliability index estimations to the VIS for the plate on elastic foundation.

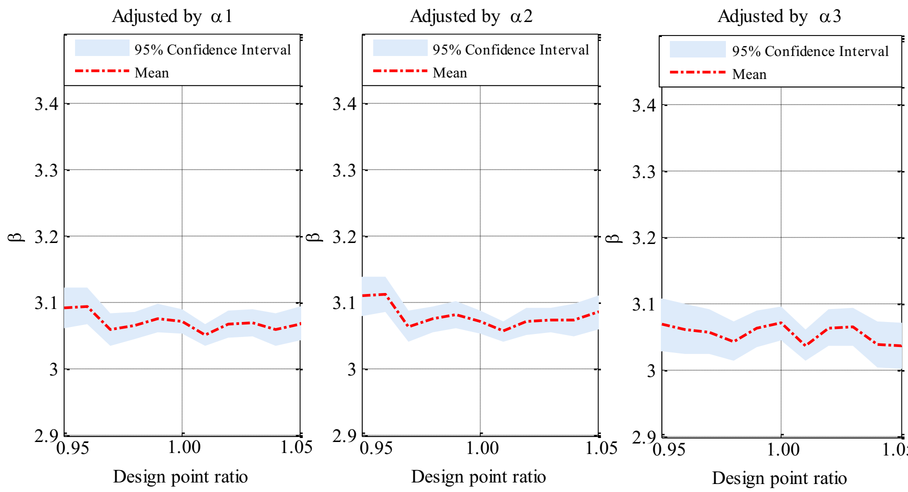

Fig. 3. Sensitivity of the reliability index estimations to variations in the location of the design point for the plate on elastic foundation.

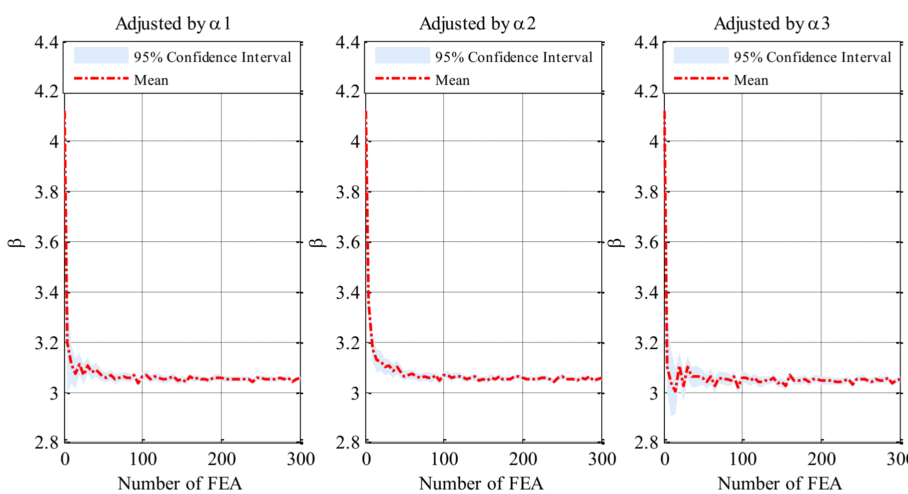

Fig. 4. Effect of sampling size on the reliability index estimations for the plate on elastic foundation.

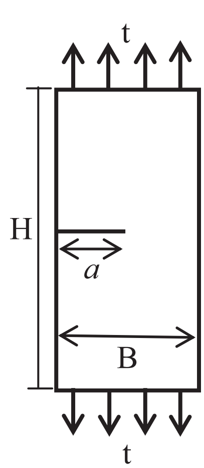

Fig. 5. Single edge notch test specimen.

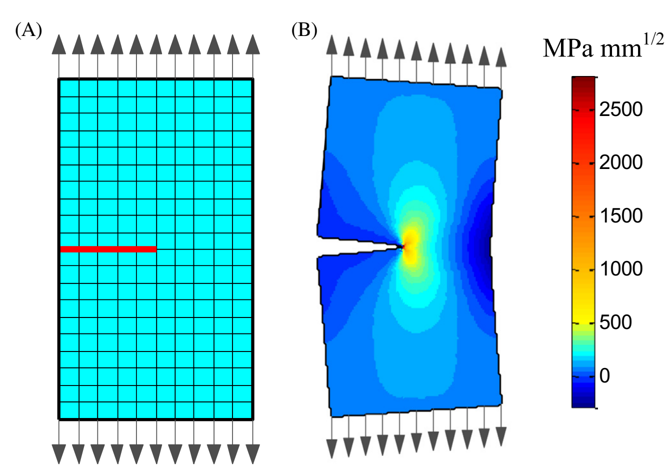

Fig. 6. FE model of the single edge notch test specimen: (A) Initial model; (B) Model showing deformation in the Y-direction.

scaled CPU time differs between the FE models with the coarse (10 × 20 elements) and fine (200 × 400 elements) mesh densities by a factor of ~3076. This significant difference in computational cost makes the application of the FE model with the fine mesh density (200 × 400 elements) impractical for reliability analysis.

The descriptive statistics of the basic random variables for the single edge notch test specimen are presented in Table 5. The performance function of the problem is given by:

Table 4
Effect of mesh density on the stress intensity factor estimations for the single edge notch test specimen. Note that the results are presented at mean values.

FE model	K_I (MPa mm$^{1/2}$)	ε_{Model}	Scaled CPU time
10 × 20	1726.823	11%	1
50 × 100	1893.006	2.4%	10.2
100 × 200	1915.734	1.27%	131.8
200 × 400	1927.255	0.67%	3076.4
Analytical function	1940.358	0.00	–

Table 5
Descriptive statistics of the basic random variables for the single edge notch test specimen.

Variable	Mean	Distribution	Standard deviation
t (MPa)	100	Normal	10
a (mm)	15	Normal	0.5
KIC (MPa mm$^{1/2}$)	3500	Normal	350

$$G = K_{IC} - K_{eq}, \tag{30}$$

where K_I is the critical stress intensity factor and K_{eq} is the existing stress intensity factor.

To evaluate the reliability of the single edge notch test specimen using FEM, the FE model with the coarse mesh density (10 × 20 elements) is selected to estimate P_f^{Coarse}. The value of P_f^{Coarse} is then refined using the FE model with the fine mesh density (200 × 400 elements) based on 70 samples and the results are tabulated in Table 6. It can be observed that the ACVAT developed in this study estimates the reliability index efficiently with high accuracy and significant reduction in computational cost.

This problem is also solved for different C.O.Vs (0.1, 0.2, 0.3, 0.4, 0.5, and 0.6) and the results are presented in Table 7. The results of the reliability analysis obtained from the ACVAT are compared with those determined from MCS using the analytical functions (Eqs. (28) and (29)).

Table 7 shows that the initial reliability index computed using the FE model with the coarse mesh density is regulated properly using the ACVAT. In general, there is good agreement between the reliability indices computed using the ACVAT adjusted by different correction factors (α_1, α_2, and α_3) and those determined from analytical functions.

3.3. Three-point bend test specimen

The applicability of the ACVAT for FEM-based reliability analysis is also demonstrated using a three-point bend test specimen. The three-point bend test specimen consists of a rectangular plate (length × width: 120 mm × 30 mm) with a crack (length: 15 mm), as shown in Fig. 7. The plate is subjected to a point load at the center. The stress intensity factor of the problem is given by [38]:

$$K_I = \frac{6PH}{4B^2}\sqrt{\pi a}F\left(\frac{a}{B}\right), \tag{31}$$

where the numerical value for F is given by:

$$F\left(\frac{a}{B}\right) = \frac{1}{\sqrt{\pi}} \cdot \frac{1.99 - \left(\frac{a}{B}\right)\left(1-\left(\frac{a}{B}\right)\right)\left(2.15 - 3.93\left(\frac{a}{B}\right) + 2.7\left(\frac{a}{B}\right)^2\right)}{\left(1+2\left(\frac{a}{B}\right)\right)\left(1-\left(\frac{a}{B}\right)\right)^{\frac{3}{2}}}. \tag{32}$$

Note that Eq. (32) is valid for $\frac{H}{B} = 4$.

The three-point bend test specimen is modeled using FEM, as shown in Fig. 8, and the results are compared with those computed using the analytical function (Eq. (31)), as shown in Table 8. It can be seen that the computational cost for the FE model with fine mesh density (100 × 400 elements) is 924 times higher than the computational cost of the FE model with coarse mesh density (10 × 40 elements). The difference in stress intensity factor between the FE model with coarse mesh density and the analytical function at mean point is only 12%.

The descriptive statistics of the basic random variables for the three-point bend test specimen are presented in Table 9. By using Eq. (30) and the proposed ACVAT, the reliability results are obtained and presented in Table 10.

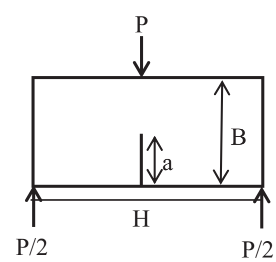

Fig. 7. Three-point bend test specimen.

Table 6
FEM-based reliability results for the single edge notch test specimen.

Parameters	Reliability results					
	5000 coarse mesh FE analyses	5000 fine mesh FE analyses	Analytical function (Eqs. {(28)–(30)})	ACVAT: Coarse mesh results + 70 fine mesh FE analyses		
				Adjusted by α_1	Adjusted by α_2	Adjusted by α_3
$\bar{\beta}$	4.317	Unfeasible	3.569	3.594	3.586	3.652
Variance				0.0079	0.0083	0.0161

Table 7
Effect of C.O.V. of the random variables on the reliability index estimations for the single edge notch test specimen.

		C.O.V					
		0.1	0.2	0.3	0.4	0.5	0.6
Reliability index	Coarse mesh FE analyses	4.317	2.381	1.454	1.199	1.037	0.675
	Adjusted by α1	3.594	1.907	1.268	0.933	0.739	0.685
	Adjusted by α2	3.586	1.923	1.235	0.994	0.797	0.635
	Adjusted by α3	3.652	1.986	1.222	1.013	0.808	0.633
	Analytical function	3.573	1.902	1.282	0.966	0.776	0.646

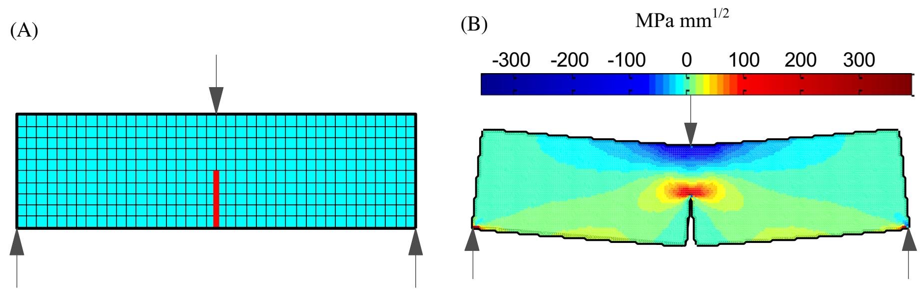

Fig. 8. FE model of the three-point bend test specimen: (A) Initial model; (B) Model showing deformation in the X-direction.

Table 8
Effect of mesh density on the stress intensity factor estimations for the three-point bend test specimen. Note that the results are presented at mean values.

FE model	K_I(MPa mm$^{1/2}$)	ε_{Model}	Scaled CPU time
10 × 40	342.098	12.03%	1
30 × 120	370.814	4.65%	10.1
50 × 200	376.837	3.10%	62.5
100 × 400	381.394	1.93%	923.8
Analytical function	388.883	0.00	–

Table 9
Descriptive statistics of the basic random variables for the three-point bend test specimen.

Variable	Mean	Distribution	Standard deviation
P (N)	200	Normal	15
a (mm)	15	Normal	1
KIC (MPa mm$^{1/2}$)	700	Normal	90
T (mm)	1	Lognormal	0.05

T: Plate thickness.

It can be seen that the results in Table 10 are similar to those for the single edge notch test specimen (Table 6). This proves that the ACVAT developed in this study (which is a combination of FE models with coarse and fine mesh densities) is capable of estimating the reliability index with a high degree of accuracy while simultaneously reducing the computational cost of the FEM-based reliability analysis by almost three orders of magnitude. In addition, the low variance obtained after performing 100 independent analyses using the proposed approach verifies that the ACVAT provides reliable estimations of the safety of the three-point bend test specimen.

3.4. Nonlinear simply supported beam

In this example, the reliability of a nonlinear simply supported beam is evaluated using the ACVAT, as shown in Fig. 9. It is assumed that the simply supported beam is made from a hypoelastic material that behaves according to the following stress-strain relationship [39]:

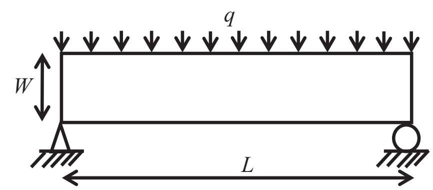

Fig. 9. Nonlinear simply supported beam.

$$\frac{\sigma_e}{\sigma_0} = \begin{cases} \sqrt{\frac{1+n^2}{(n-1)^2} - \left(\frac{n}{n-1} - \frac{\varepsilon_e}{\varepsilon_0}\right)^2} - \frac{1}{n-1}, & \varepsilon_e \leqslant \varepsilon_0 \\ \left(\frac{\varepsilon_e}{\varepsilon_0}\right)^{1/n}, & \varepsilon_e \geqslant \varepsilon_0 \end{cases} \tag{33}$$

The stress-strain curve of the hypoelastic material is shown in Fig. 10.

In order to evaluate the reliability of the nonlinear simply supported beam, the span (L), width (W), distributed load (q), and parameters related to the stress-strain curve (σ_0, ε_0, n) are treated as random variables. The performance function of the problem is defined as follows:

$$G = u_{allowable} - u_m. \tag{34}$$

Here, $u_{allowable}$ is the allowable deflection and u_m is the maximum deflection obtained from FE analysis. In this study, $u_{allowable}$ is 15 cm. The descriptive statistics of the random variables for the nonlinear simply supported beam are tabulated in Table 11.

The effect of mesh density on the maximum deflection estimations are presented in Table 12. The results show that the percentage difference in maximum deflection estimation between the FE models with coarse (30 × 5 elements) and fine (60 × 10 elements) mesh densities is 8.1%. However, the FE model with the fine mesh density consumes more CPU time by a factor of 4.3.

The results of the FEM-based reliability analysis are presented in Table 13. It can be seen that the reliability index estimated from 5000 fine mesh FE analyses is β = 3.36 whereas the reliability index estimated from 5000 coarse mesh FE analyses is slightly higher, where β = 3.91. It is evident that the ACVAT developed in this study is capable of estimating the reliability index accurately with fewer fine mesh FE analyses, which significantly reduces computational cost. Based on the accuracy of the estimations and significant reduction in computational cost, it can be deduced that the pro-

Table 10
FEM-based reliability results for the three-point bend test specimen.

Parameters	Reliability results					
	5000 coarse mesh FE analyses	5000 fine mesh FE analyses	Analytical function (Eqs. (30)–(32))	ACVAT: Coarse mesh results + 70 fine mesh FE analyses		
				Adjusted by α_1	Adjusted by α_2	Adjusted by α_3
$\bar{\beta}$	3.379	Unfeasible	2.961	2.921	2.942	2.849
Variance				0.0525	0.0456	0.0684

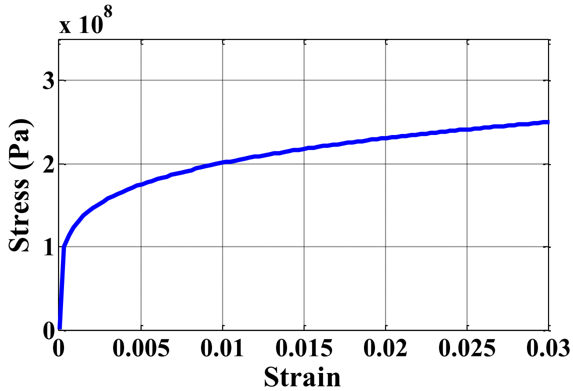

Fig. 10. Stress-strain curve of the hypoelastic material.

Table 11
Descriptive statistics of the basic random variables for the nonlinear simply supported beam.

Variable	Mean	Distribution	C.O.V
L (m)	3	Normal	0.02
W (m)	0.5	Normal	0.02
q (kN/m)	100	Normal	0.20
σ_0 (MPa)	250	Normal	0.05
ε_0	0.03	Normal	0.05
n	5	Normal	0.05

Table 12
Effect of mesh density on the maximum displacement estimations for the nonlinear simply supported beam. Note that the results are presented at mean values.

FE model	u_m (cm)	ε_{Model}	Scaled CPU time
30 × 5	4.54	8.10%	1
42 × 7	4.71	4.65%	1.73
54 × 9	4.86	0.40%	2.88
60 × 10	4.94	0.00%	4.33

Table 13
FEM-based reliability results for the nonlinear simply supported beam.

Parameters	Reliability results				
	5000 coarse mesh FE analyses	5000 fine mesh FE analyses	ACVAT: Coarse mesh results + 70 fine mesh FE analyses		
			Adjusted by α_1	Adjusted by α_2	Adjusted by α_3
β	3.91	3.36	3.31	3.33	3.30

posed approach is capable of estimating the reliability of nonlinear engineering structures.

4. Conclusions

From a deterministic point of view, mesh convergence analysis provides information regarding the model selected for reliability analysis and the errors associated with the estimations. Based on the results of the mesh convergence analysis, a suitable FE model is selected to estimate the reliability of engineering structures and small errors in the estimations are considered as acceptable. In this study, it is shown that errors arising from mesh density inadequacy have a significant effect on estimating the reliability (and hence, safety) of engineering structures. Hence, in order to estimate the reliability of engineering structures with a high degree of accuracy, a mesh density with an error of ~1% should be used in FEM-based reliability analysis. However, it is impractical to conduct FE simulations with very fine mesh densities to evaluate the reliability of most engineering structures due to the large computational cost incurred, especially when reliability analysis is integrated with design optimization. Hence, a new method (named ACVAT) is proposed in this study to address this issue, where linear CVT is integrated with FEM for reliability analysis and the capability of the method is enhanced by IS. In addition, two known correction factors (α_1 and α_2) are adjusted to improve the efficiency of the simulations and a new correction factor (α_3) is proposed in this study to achieve a converged solution efficiently using the ACVAT. The proposed approach is used for FEM-based reliability analysis of four engineering structures as illustrative examples and the results show that there is significant reduction in the computational cost using the ACVAT. The ACVAT requires the combination of FE models with coarse and fine mesh densities in order to estimate the reliability of engineering structures with high accuracy and simultaneously reduce computational processing time. Based on the illustrative examples presented in this paper, the accuracy of the ACVAT is only guaranteed for linear and mildly nonlinear problems and it may be impractical for highly nonlinear FE problems.

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