OPTOELECTRONICS (I)

Chapter 2:
Light Propagation

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Review of the electromagnetic theory of light

1) Maxwell's equations: wave equation

Light is, according to classical theory, the flow of electromagnetic (EM) radiation through free space or through a medium in the form of **electric** and **magnetic** **fields**. Although electromagnetic radiation covers an extremely wide range, from **gamma rays** to **long radio waves**, the term "**light**" is **restricted** to the part of the electromagnetic spectrum that goes from the vacuum **ultraviolet** to the **far infrared**. This part of the spectrum is also called ***optical range***.

EM radiation propagates in the form of two mutually **perpendicular** and coupled **vectorial waves**: the **electric field** **E**(**r**, t) and the **magnetic field** **H**(**r**, t).

These **two** vectorial magnitudes depend on the **position** (**r**) and **time** (t)

1.1 Material Categorize

▪از نظر وابستگی ضرایب اساسی شان (ε و μ) به **دامنه** میدانهای H, E:

۱. خطی (Linear) ۲. غیرخطی (ضرایب ε و μ به صورت تابعی از H, E خواهند بود.)

▪از نظر وابستگی ضرایب اساسی شان (ε و μ) به **موقعیت** مختصاتی و مکانی:

۱. همگن (Homogeneous) ۲. غیرهمگن (ضرایب ε و μ به صورت تابعی از مکان سه بعدی x,y,z خواهند بود.)

▪از نظر وابستگی ضرایب اساسی شان (ε و μ) به **جهت اعمال** میدانهای H, E:

۱. همسانگرد (Isotropic) ۲. ناهمسانگرد (anisotropic) (ضرایب ε و μ به صورت تانسوری خواهند بود.)

▪از نظر وابستگی ضرایب اساسی شان (ε و μ) به **فرکانس**:

۱. غیرپاشنده (nondispersive) ۲. پاشنده (Dispersive) (ضرایب ε و μ تابعی از فرکانس میدان اعمالی خواهند بود.)

$$\overline{D}(\overline{r}) = \varepsilon_0 \varepsilon(\overline{r}) \overline{E}(\overline{r})$$

$$\overline{B} = \mu_0 \overline{H}$$

2. Classical electromagnetism

2.1 Electrostatics:

E: electric field
D: displacement vector field

B: magnetic flux density
H: magnetic field

***V*:** Potential
ρ: charge density

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \varepsilon_r \quad (57)$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}.$$

$$\nabla \cdot \mathbf{B} = 0 \quad (58)$$

$$\mathbf{H} = \mathbf{B} / \mu_0 \mu_r$$

$$\mathbf{E} = -\nabla V$$

Gauss's law: (Stoke's theorem)

$$\int_V \nabla \cdot \mathbf{E}\, dV = \oint_S \mathbf{E} \cdot \mathbf{n}^{\sim} dS = \int_V (\rho / \varepsilon_0 \varepsilon_r) dV \quad (59)$$

2. Classical electromagnetism

2.1 Electrostatics:

Capacitance:

$$C = \frac{Q}{V} \quad (60)$$

$$C = \frac{Q}{V} = \frac{\rho A}{\rho d/\varepsilon_0 \varepsilon_r} = \frac{\varepsilon_0 \varepsilon_r A}{d} \quad (61)$$

stored energy:

$$\Delta E = \int_{t'=-\infty}^{t'=t} CV \frac{dV}{dt'} dt' = \int_{V'=0}^{V'=V} CV' dV' = \frac{1}{2} CV^2 \quad (62)$$

$$\Delta E = \frac{1}{2} CV^2 \quad (63)$$

stored *energy density:* ΔU

1. energy stored per unit volume in the electric field:

$$\Delta U = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} \quad (64)$$

energy stored per unit volume in a magnetic field:

$$\Delta U = \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \quad (65)$$

Inductance

$$L = \frac{1}{I} \int_S \mathbf{B} \cdot \mathbf{n}^{\sim} dS \quad (66)$$

2. Classical electromagnetism

Gauss's law:

$$\int_V \nabla \cdot \mathbf{E}\, dV = \oint_S \mathbf{E} \cdot \tilde{\mathbf{n}}\, dS = \int_V (\rho/\varepsilon_0\varepsilon_r)\, dV \quad (67)$$

electric flux:

$$E_r = \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2} \quad (68)$$

Potential:

$$V = -\int_{r_2}^{r_1} E_r\, dr = -\int_{r_2}^{r_1} \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}\, dr = \frac{Q}{4\pi\varepsilon_0\varepsilon_r}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad (69)$$

Capacitance:

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0\varepsilon_r}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \quad (70)$$

$$C = 4\pi\varepsilon_0\varepsilon_r r_1 \quad (71)$$

charging energy:

$$\Delta E = \int_{t'=-\infty}^{t'=1} CV \frac{dV}{dt'}\, dt' = \int_{V'=0}^{V'=V} CV'\, dV' = \frac{1}{2} CV^2 = \frac{Q^2}{2C} \quad (72)$$

$$\Delta E = \frac{e^2}{2C} = \frac{e^2}{8\pi\varepsilon_0\varepsilon_r r_1} \quad (73)$$

2. Classical electromagnetism: (2.2 Electrodynamics)

Classical electrodynamics describes the spatial and temporal behavior of electric and magnetic fields.

Plane waves can be represented *spatially as:*

$$\sin(kx) = \frac{1}{2i}(e^{ikx} - e^{-ikx}) \qquad (74)$$

$$\cos(kx) = \frac{1}{2}(e^{ikx} + e^{-ikx}) \qquad (75)$$

$$e^{ikx} = \cos(kx) + i\sin(kx) \qquad (76)$$

Plane waves can be represented *temporally by:*

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t) \qquad (77)$$

plane wave: $$Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad (78)$$

A: amplitude of the wave,
$K=2\pi/\lambda$:**wave vector of magnitude**,
$\omega=2\pi f$: angular frequency,
$f=1/\tau$: frequency,
τ: periode

2.2 Electrodynamics

Table 1.1 *Maxwell equations*

$\nabla \cdot \mathbf{D} = \rho$	Coulomb's law
$\nabla \cdot \mathbf{B} = 0$	No magnetic monopoles
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	Modified Ampère's law

D :displacement vector field

E: electric field, or electric flux density

χ_e: electric susceptibility

P: electric polarization field

H and B: The magnetic field vector, or the magnetic flux density

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_0 \varepsilon_r \boldsymbol{E} = \varepsilon_0 (1 + \chi_e) \boldsymbol{E} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} = \mu_0 \mu_r \boldsymbol{H} = \mu_0 (1 + \chi_m) \boldsymbol{H}$$

$$= \mu_0 (\boldsymbol{H} + \boldsymbol{M})$$

μ: permeability,

μ_r: relative permeability,

χ_m : magnetic susceptibility,

M: magnetization

2.2 Electrodynamics

divergence theorem:

$$\int_V \nabla \cdot \mathbf{a}\, d^3 r = \int_S \mathbf{a} \cdot \mathbf{n}^{\sim}\, dS \tag{84}$$

S. Stokes' theorem:

$$\int_S (\nabla \times \mathbf{a}) \cdot \mathbf{n}^{\sim}\, dS = \oint_C \mathbf{a} \cdot d\mathbf{l} \tag{85}$$

V: volume
n~: unit-normal vector to the surface S
J: current density
ρ: charge density

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{86}$$

$$0 = \nabla \cdot \mathbf{J} + \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{87}$$

$$0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \tag{88}$$

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot \mathbf{n}^{\sim} dS = \int_S \mathbf{J} \cdot \mathbf{n}^{\sim} dS = I \tag{89}$$

2.3 Light propagation in a dielectric medium

In the **dielectric**, current density $\mathbf{J} = 0$ because the dielectric has no mobile charge, and if $\mu_r = 1$ at optical frequencies then $\mathbf{H} = \mathbf{B}/\mu_0$.

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu_0 \frac{\partial^2}{\partial t^2}\mathbf{D} \qquad (90)$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial^2}{\partial t^2}\mathbf{D} \qquad (91)$$

$$\nabla^2 \mathbf{E} = \mu_0 \frac{\partial^2}{\partial t^2}\mathbf{D} \qquad (92)$$

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) = \frac{\partial^2}{\partial t^2}\mu_0 \varepsilon \mathbf{E}(\mathbf{r}, t) \qquad (93)$$

$$\nabla^2 \mathbf{E}(\mathbf{r}, \omega) = -\omega^2 \mu_0 \varepsilon_0 \varepsilon_r(\omega)\mathbf{E}(\mathbf{r}, \omega) \qquad (94)$$

wave equation:

$$\nabla^2 \mathbf{E}(\mathbf{r}, \omega) = \frac{-\omega^2}{c^2}\varepsilon_r(\omega)\mathbf{E}(\mathbf{r}, \omega) \qquad (95)$$

Solution: plane waves.

2.3 Light propagation in a dielectric medium

If $\varepsilon_r(\omega)$ is real and positive, the solutions to this wave equation for an electric field propagating in an isotropic medium are just plane waves. The **speed** of wave propagation is $c/n_r(\omega)$, where $n_r(\omega)=[\varepsilon_r(\omega)]^{1/2}$ is the ***refractive index of the material***. *In the more* general case, when relative permeability $\mu_r \neq 1$, the refractive index is:

$$n_r(\omega) = \sqrt{\varepsilon_r(\omega)}\sqrt{\mu_r(\omega)} = \frac{\sqrt{\varepsilon(\omega)}\sqrt{\mu(\omega)}}{\sqrt{\varepsilon_0\mu_0}} \qquad (96)$$

If one of either ε or μ is negative, refractive index is imaginary and electromagnetic waves cannot propagate. It is common for metals to have negative values of ε.

In a metal, free electrons can collectively oscillate at a long-wavelength natural frequency called the **plasma frequency**, $\omega_p = (ne^2/\varepsilon_0 m)^{1/2}$.

$\varepsilon_r(\omega) = 1-\omega^2_p/\omega^2$: a good approximation for a metal at long wavelengths.

If $\omega \gg \omega_p$: ε=positive ,and electromagnetic waves can propagate through the metal.

For $\omega \ll \omega_p$: ε=negative , *n* is imaginary, waves cannot propagate in the metal and are reflected.

why bulk metals are usually not transparent to electromagnetic radiation of frequency less than ω_p?

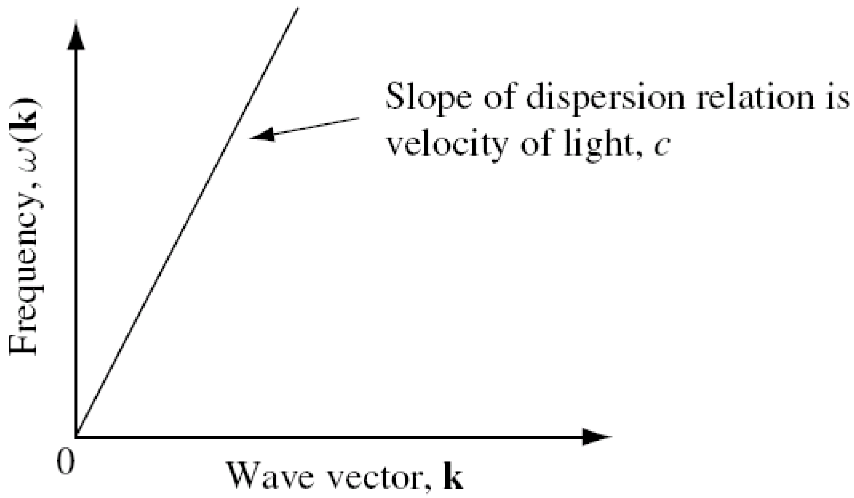
2.3. Light propagation in a dielectric medium

In a homogeneous dielectric medium: $\mu_r = 1$ and $\varepsilon(\omega)=\varepsilon_0\varepsilon_r = \varepsilon_0(\varepsilon'_r(\omega)+ \varepsilon''_r(\omega))$ where $\varepsilon'_r(\omega)$ and $\varepsilon''_r(\omega)$ are the real and imaginary parts. In this situation:

$$\mathbf{E}(\mathbf{r},\omega) = \mathbf{E}_0(\omega)e^{i\mathbf{k}(\omega)\cdot\mathbf{r}} = \mathbf{E}_0(\omega)e^{i(k'(\omega)+ik''(\omega))\mathbf{k}^{\sim}\cdot\mathbf{r}} \quad (97)$$

$$n_r(\omega) = \sqrt{\frac{1}{2}(\varepsilon'_r(\omega) + \sqrt{\varepsilon'^2_r(\omega) + \varepsilon''^2_r(\omega)})} \quad (98)$$

Fig. 1.17 Dispersion relation for an electromagnetic wave in **free space**. The slope of the line is the velocity of light.



2.3 Light propagation in a dielectric medium

For the case: $k''(\omega) = 0$ and $\mu_r = 1$, the refractive index is just $n_r(\omega) = [\varepsilon'_r(\omega)]^{1/2}$, and we have a simple oscillatory solution with no spatial decay in the electric and magnetic field vector:

$$\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0 e^{-i\omega t} e^{i\mathbf{k}(\omega)\cdot\mathbf{r}} \quad (99)$$

$$\mathbf{H}(\mathbf{r}, \omega) = \mathbf{H}_0 e^{-i\omega t} e^{i\mathbf{k}(\omega)\cdot\mathbf{r}} \quad (100)$$

Maxwell's equations in free space:

$$\nabla \cdot \mathbf{D} = 0 \quad (101)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (102)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (103)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad (104)$$

The first two equations are divergence equations that require that **$\mathbf{k} \cdot \mathbf{E} = 0$ and $\mathbf{k} \cdot \mathbf{B} = 0$.** This means that **E and B** are **perpendicular (transverse)** to the **direction of propagation k~.**

in free space:

$$\nabla \times \mathbf{E}_0 e^{-i\omega t} e^{i\mathbf{k}(\omega)\cdot\mathbf{r}} = -\mu_0 \frac{\partial}{\partial t} \mathbf{H}_0 e^{-i\omega t} e^{i\mathbf{k}(\omega)\cdot\mathbf{r}} \quad (105)$$

$$i\mathbf{k} \times \mathbf{E}_0 e^{-i\omega t} e^{i\mathbf{k}(\omega)\cdot\mathbf{r}} = i\omega \mu_0 \mathbf{H}_0 e^{-i\omega t} e^{i\mathbf{k}(\omega)\cdot\mathbf{r}} \quad (106)$$

$$i\mathbf{k} \times \mathbf{E} = i\omega \mu_0 \mathbf{H} \quad (107)$$

2.3 Light propagation in a dielectric medium

Using the fact that the dispersion relation for plane waves in free space is $\omega = ck$ and the speed of light is $c = 1/[\varepsilon_0\mu_0]^{1/2}$, leads us directly to:

$$\mathbf{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\mathbf{k}^{\sim} \times \mathbf{E} \quad (108)$$

$$\mathbf{B}c = \mathbf{k}^{\sim} \times \mathbf{E},$$

where $\mathbf{k}^{\sim} = \mathbf{k}/|\mathbf{k}|$ is the unit vector for $\mathbf{k}$.

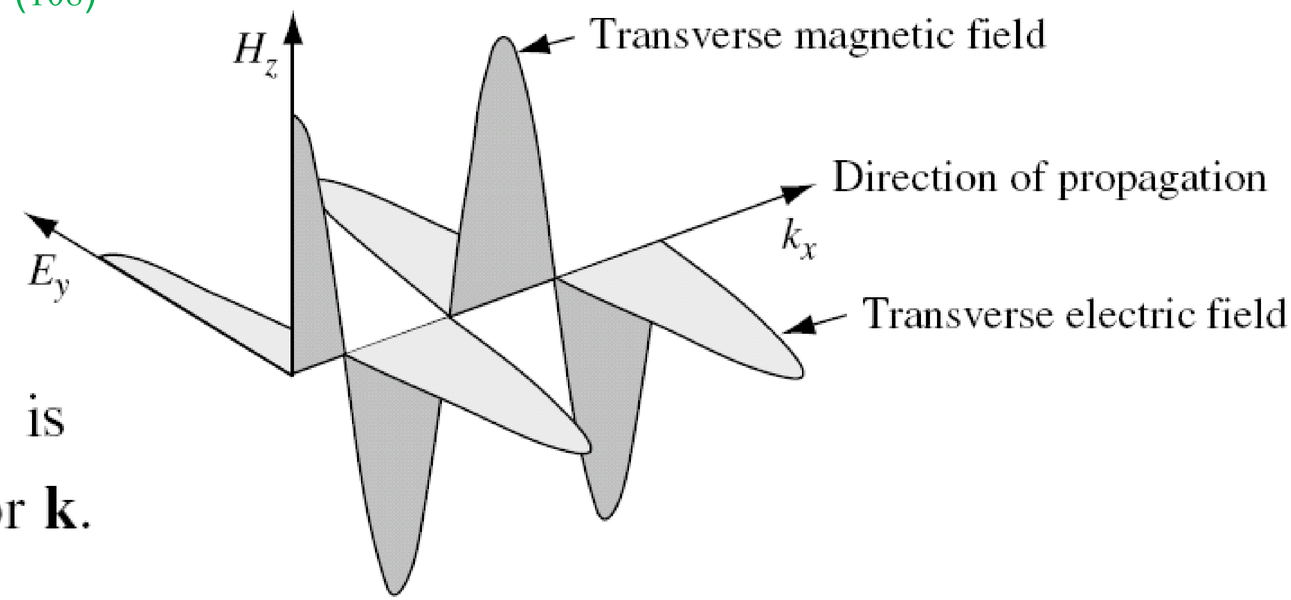


Fig. 1.18 Illustration of transverse magnetic field H_z and electric field E_y of a plane wave propagating in free space in the x direction.

2.3 Light propagation in a dielectric medium

Oscillating transverse electromagnetic **waves** can **decay** in **time** and in **space**.

$$\mathbf{E}(t) = \mathbf{y}\tilde{}\,|\mathbf{E}_0|\sin(\omega t)e^{-\gamma t}$$

$$\mathbf{E}(x) = \mathbf{y}\tilde{}\,|\mathbf{E}_0|\cos(kx)e^{-\gamma x}$$

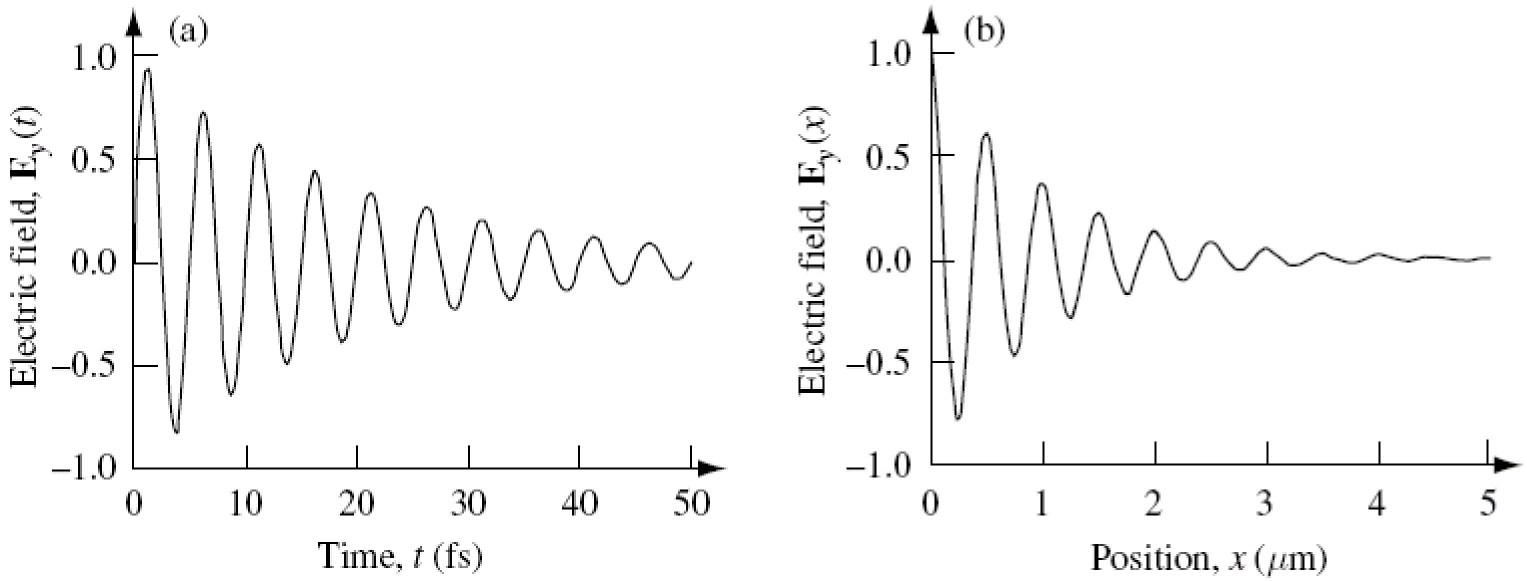


Fig. 1.19 (a) Illustration of **temporal decay** of an oscillating electric field.
(b) Illustration of **spatial decay** of an oscillating electric field.

2.4 Power and momentum in an electromagnetic wave

The power in an electromagnetic wave can be obtained by considering the response of a test **charge e** moving at **velocity** v in an external **electric field E.** The rate of work or power is just $e\mathbf{v}\cdot\mathbf{E}$, where $e\mathbf{v}$ is a current. The total power in a given volume is:

$$\int_{\text{Volume}} d^3r\,\mathbf{J}\cdot\mathbf{E} = \int_{\text{Volume}} \left(\mathbf{E}\cdot(\nabla\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right) d^3r \qquad (109)$$

Because:

$$\nabla\times\mathbf{H} = \mathbf{J} + \partial\mathbf{D}/\partial t$$

From:

$$\mathbf{E}\cdot(\nabla\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{H}) \text{ and } \nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t,$$

$$\int_{\text{Volume}} d^3r\,\mathbf{J}\cdot\mathbf{E} = -\int_{\text{Volume}} \left(\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right) d^3r \qquad (110)$$

Or on different form:

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} = -\mathbf{J}\cdot\mathbf{E} - \nabla\cdot(\mathbf{E}\times\mathbf{H}) \qquad (111)$$

2.4 Power and momentum in an electromagnetic wave

From (64), (65), (111):

$$\Delta U = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$$

$$\Delta U = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$$

$$\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{J}\cdot\mathbf{E} - \nabla\cdot(\mathbf{E}\times\mathbf{H})$$

The total energy density:

$$U = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) \quad (112)$$

$$\frac{\partial U}{\partial t} = -\mathbf{J}\cdot\mathbf{E} - \nabla\cdot\mathbf{S} \quad (113)$$

***S*: Poynting vector:**

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \quad (114)$$

The Poynting vector is the energy flux density in the electromagnetic field.

In free space, The total energy density:

$$U = \frac{|S|}{c} \quad (115)$$

2.4 Power and momentum in an electromagnetic wave

In free space:

$$U = \frac{|S|}{c} \quad (115)$$

$$\mathbf{E}(\mathbf{r},\omega) = \mathbf{E}_0 e^{-i\omega t} e^{i\mathbf{k}(\omega)\cdot\mathbf{r}} \quad (99)$$

$$\mathbf{H} = \sqrt{\varepsilon_0/\mu_0}\,\mathbf{k}^{\sim} \times \mathbf{E} \quad (108)$$

$$\mathbf{H}(\mathbf{r},\omega) = \mathbf{H}_0 e^{-i\omega t} e^{i\mathbf{k}(\omega)\cdot\mathbf{r}} \quad (100)$$

***S:* Poynting vector:**

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\mathbf{E} \times \mathbf{k}^{\sim} \times \mathbf{E} \quad (116)$$

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$$\mathbf{S} = \sqrt{\frac{\varepsilon_0}{\mu_0}}((\mathbf{E}\cdot\mathbf{E})\mathbf{k}^{\sim} - (\mathbf{E}\cdot\mathbf{k}^{\sim})\mathbf{E}) \quad (117)$$

$(\mathbf{E}\cdot\mathbf{K}^{\wedge}) = 0$ →

$$\mathbf{S} = \sqrt{\frac{\varepsilon_0}{\mu_0}}(\mathbf{E}\cdot\mathbf{E})\mathbf{k}^{\sim} \quad (118)$$

Defining the ***impedance of free space:***

$$Z_0 \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 376.73\,\Omega \qquad Z_0 = 120 \times \pi\,\Omega \quad (119)$$

→

$$\mathbf{S} = \frac{(\mathbf{E}\cdot\mathbf{E})}{Z_0}\mathbf{k}^{\sim} \quad (120)$$

For **monochromatic plane waves** propagating in the **x direction**, the **Poynting vector:**

$$\mathbf{S} = \frac{|\mathbf{E}_0|^2}{Z_0}(\cos^2(k_x x - \omega t + \Delta_{\text{phase}}))\mathbf{k}^{\sim} \quad (121)$$

$$\langle \mathbf{S} \rangle = \frac{|\mathbf{E}_0|^2}{2Z_0}\mathbf{k}^{\sim} \quad (122)$$

2.4 Power and momentum in an electromagnetic wave

Momentum: p

Electromagnetic waves carry not only **energy**, but also **momentum**.

The classical Lorentz force on a test charge **e** moving at velocity **v** is:

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (123)$$

$$\mathbf{F} = d\mathbf{p}/dt \quad \Rightarrow \quad \mathbf{p} = \frac{\mathbf{E} \times \mathbf{H}}{c^2} = \frac{\mathbf{S}}{c^2} \qquad (124)$$

momentum can be expressed in terms of the energy density as:

$$\mathbf{p} = \frac{U}{c}\mathbf{k}^{\sim} \qquad (125)$$

The magnitude of the momentum is just:

$$|\mathbf{p}| = \frac{1}{c}\frac{|\mathbf{S}|}{c} = \frac{U}{c} \qquad (126)$$

$$\Rightarrow \quad p = \frac{U}{c} \qquad (127)$$

2.5 Choosing a potential

In general, Maxwell's equations allow electric and magnetic fields to be described in terms of a scalar potential **V(r, t)** **and a vector potential A(r, t).**

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0 \qquad \nabla \cdot \mathbf{B} = 0, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla \times \mathbf{A} \quad (128)$$

or:

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \quad (129)$$

Since the curl of the gradient of any scalar field is zero, we may equate the last equation with the gradient of a scalar field, V, where:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\Delta V \quad (130)$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla V(\mathbf{r}, t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t) \quad (131)$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \quad (132)$$

3. Maxwell's equation in a material medium

Maxwell's equations form a set of **four coupled equations** involving the electric field vector and the magnetic field vector of the **light**, and are **based on** experimental evidence.
Two of them are scalar equations, and the other two are vectorial.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla . \mathbf{D} = \rho$$

$$\nabla . \mathbf{B} = 0$$

دانسیته فلوی الکتریکی

$\mathbf{D}(\mathbf{r},t)$: ***electric displacement vector***

$\mathbf{B}(\mathbf{r},t)$: ***magnetic flux density vector***

دانسیته فلوی مغناطیسی

$\rho(\mathbf{r},t)$: ***charge density***

$\mathbf{J}(\mathbf{r},t)$: ***current density vector***

No free pole

If in the **medium** there are **no free electric charges**, **which is the most common situation** in **optics**, Maxwell's equations simplify in the form: $\nabla . \mathbf{D} = 0$

3. Maxwell's equation in a material medium

These relations are called *constitutive relations*, and **depend on** the **electric** and **magnetic properties** of the considered **medium**.

For a linear, **homogeneous** and **isotropic** medium, the *constitutive relations* are given by:

$$\mathbf{D} = \varepsilon \mathbf{E} \quad , \quad \mathbf{B} = \mu \mathbf{H} \quad , \quad \mathbf{J} = \sigma \mathbf{E}$$

ε is the **dielectric permittivity**, μ is the **magnetic permeability** and σ is the **conductivity** of the medium.

نفوذ ناپذیری

- A **homogeneous medium** implies that the **optical constants** of the medium ε, μ and σ are not dependent of the position vector **r**.

- In an **isotropic medium** these optical constants are scalar magnitudes and **independent** of the **direction** of the vectors **E** and **H**, implying that the vectors **D** and **J** are **parallel** to the electric field **E**, and the vector **B** is **parallel** to the magnetic field **H**.

3. Maxwell's equation in a material medium

- By using the constitutive relations for a linear, homogenous and isotropic medium, Maxwell's equations can be written in terms of the electric field E and magnetic field H only

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla . \mathbf{E} = 0$$

$$\nabla . \mathbf{H} = 0$$

3.1 Wave equation

$$\nabla \times (\nabla \times y) = \nabla(\nabla . y) - \nabla^2 y$$

$$\nabla \times (\nabla \times E) = \nabla(\nabla . E) - \nabla^2 E, \qquad \nabla \bullet (E) = \nabla \bullet (D/\varepsilon) = 0$$

$$\nabla \times (\nabla \times E) = -\nabla \times (\frac{\partial B}{\partial t}) = -\nabla \times (\mu \frac{\partial H}{\partial t}) \Rightarrow \nabla(\nabla . E) - \nabla^2 E = \mu \frac{\partial(\nabla \times H)}{\partial t}$$

$$\nabla^2 E = \mu \frac{\partial\left(\partial D / \partial t\right)}{\partial t} \Rightarrow$$

$$\nabla^2 E = \varepsilon\mu \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 H = \varepsilon\mu \frac{\partial^2 H}{\partial t^2}$$

که همان معادله موج است

$$E \equiv E(x, y, z, t)$$
$$H \equiv H(x, y, z, t)$$

$$\nabla \Phi^2 = \frac{1}{v_p^2} \frac{\partial^2 \Phi}{\partial t^2}$$

$$v_p = \frac{1}{\sqrt{\mu_\varepsilon}} = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}}$$

3. Maxwell's equation in a material medium

- By combining adequately these **four differential equations**, it is possible to obtain **two** differential equations in partial derivatives, one for the **electric field** and another for the **magnetic field**.

$$\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{H} = \mu\sigma \frac{\partial \mathbf{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

- These two differential equations are known as ***wave equations*** for a **material medium**.

- The **solution of both equations** are not independent, because the **electric** and **magnetic fields** are related through Maxwell's equations.

3.2 Wave equation in a dielectric media

- A **perfect** dielectric medium is defined as a material in which the conductivity is $\sigma = 0$.

- In this category fall most of the **substrate materials** used for **integrated optical devices**, such as **glasses**, **ferro-electric crystals** or **polymers**, while **metals** do not belong to this category because of their high conductivity.

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{H} = \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

- Each of these two **vectorial wave equations** can be separated on **three scalar wave equations**, expressed as:

$$\nabla^2 \xi = \mu\varepsilon \frac{\partial^2 \xi}{\partial t^2}$$

3.2 Wave equation in a dielectric media

- The scalar variable $\xi(\mathbf{r}, t)$ may represent each of the six Cartesian components of either the electric and magnetic fields.

- The solution of this equation represents a wave that propagates with a speed v (*phase velocity*) given by:

$$\upsilon = \frac{1}{\sqrt{\mu\varepsilon}}$$

- For propagation in free space, and using the values for ε_0 and μ_0 we obtain:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \approx 3.00\times 10^8\,\mathrm{ms}^{-1}$$

- which corresponds to the **speed of light** in **free space** measured experimentally.
- The **speed of light** has been obtained **only** using values of **electric** and **magnetic** constants.

3.2 Wave equation in a dielectric media

- The **propagation speed** of the electromagnetic waves in a **medium** v as function of the speed of light in **free space** c,

$$v \equiv \frac{c}{n}$$

- **n** represents the ***refractive index*** of the dielectric medium.

- The *refractive index* is related with the **optical constant** of the material medium and the dielectric permittivity and the magnetic permeability of the **free space** by:

$$n \equiv \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

- In most of the materials (non-magnetic materials), and in particular in **dielectric media**, the magnetic **permeability** is very close to that of free space: $\mu \approx \mu_0$.

$$n \approx \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\varepsilon_r}$$

3.2 Wave equation in a dielectric media

- ε_r: relative dielectric permittivity (dielectric constant), defined as the relation between the dielectric permittivity of the material medium and that of the free space.

Material	Refractive index	Wavelength (nm)
Glass (BK7)	1.51	633
Glass (ZBLAN)	1.50	633
Polymer (PMMA)	1.54	633
Silica (amorphous SiO_2)	1.45	633
Quartz (SiO_2)	1.55	633
Silicon nitride (Si_3N_4)	2.10	633
Calcium fluoride (CaF_2)	1.43	633
Lithium niobate ($LiNbO_3$)	2.28 (n_o) 2.20 (n_e)	633
Silicon (Si)	3.75	1300
Gallium arsenide (GaAs)	3.4	1000
Indium phosphide (InP)	3.17	1510

Refractive indices corresponding to materials commonly used in the fabrication of integrated photonic components

3.3 Monochromatic waves

- The **time dependence** of the **electric** and **magnetic fields** within the wave equations admits solutions of the form of **harmonic functions**. **Electromagnetic waves** with such sinusoidal **dependence on** the time variable are called *monochromatic waves*, and are characterised by their **angular frequency** ω. In a general form, the electric and magnetic fields associated with a monochromatic wave can be expressed as:

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0(\mathbf{r})\cos\left[\omega t+\varphi(\mathbf{r})\right]$$

$$\mathbf{H}(\mathbf{r},t)=\mathbf{H}_0(\mathbf{r})\cos\left[\omega t+\varphi(\mathbf{r})\right]$$

- where the fields amplitudes $\mathbf{E_0}(\mathbf{r})$ and $\mathbf{H_0}(\mathbf{r})$ and the initial phase $\varphi(\mathrm{r})$ depend on the position r, but the **time dependence** is carried out only in the **cosine** argument through ωt.

3. 4 Complex Notation of Monochromatic waves

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{r})e^{+i\omega t}\right]$$

$$\mathbf{H}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{H}(\mathbf{r})e^{+i\omega t}\right]$$

✓ **E(r)** and **H(r)** denote the *complex amplitudes* of the electric and magnetic fields, respectively.

✓The electromagnetic spectrum covered by **light** (optical spectrum) ranges from frequencies of 3×10^5 Hz corresponding to the far IR, to 6×10^{15} Hz corresponding to vacuum UV, being the frequency of visible light around 5×10^{14} Hz.

✓The **average** of the Poynting vector as a function of the *complex fields amplitudes* for **monochromatic waves**

$$\langle S \rangle = \left\langle \mathrm{Re}\left[\mathbf{E}e^{+i\omega t}\right] \times \mathrm{Re}\left[\mathbf{H}e^{+i\omega t}\right] \right\rangle = \mathrm{Re}\{\mathbf{S}\}$$

3.4 Complex Notation of Monochromatic waves

- **S** has been defined as: $\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$
- **S** is called the *complex Poynting vector.*
- The **intensity** carried by a **monochromatic EM** wave should be expressed as: $I = \left|\mathrm{Re}\{\mathbf{S}\}\right|$
- In the case of monochromatic waves, Maxwell's equations using the complex fields amplitudes **E** and **H** are simplified notably (a dielectric and non-magnetic medium, $\sigma = 0$ and $\mu = \mu_0$)

$$\nabla \times \mathbf{E} = -i\mu_0\omega\mathbf{H}$$

$$\nabla \times \mathbf{H} = i\varepsilon\omega\mathbf{E}$$

$$\nabla . \mathbf{E} = 0$$

$$\nabla . \mathbf{H} = 0$$

3.4 Complex Notation of Monochromatic waves

✓ Now, if we **substitute** the **solutions** on the form of **monochromatic waves** in the **wave equation,** we obtain a new wave equation, **valid only** for **monochromatic waves**, known as the ***Helmholtz equation***:

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = 0 \qquad k \equiv \omega(\varepsilon\mu_0)^{1/2} = nk_0$$

$$\nabla^2 \mathbf{H}(\mathbf{r}) + k^2 \mathbf{H}(\mathbf{r}) = 0 \qquad k_0 \equiv \omega / c$$

✓ If the material medium is **inhomogeneous** the dielectric permittivity is **no longer** constant, but position dependent $\varepsilon = \varepsilon(\mathbf{r})$. The Helmholtz equations are **not longer** valid.

✓For a locally homogeneous medium, in which $\varepsilon(\mathbf{r})$ varies slowly for distances of $\sim 1/k$, those wave equations are **approximately** valid by now defining $k = n(\mathbf{r})k_0$, and $n(\mathbf{r}) = [\varepsilon(\mathbf{r})/\varepsilon_0]^{1/2}$.

3.5 Monochromatic plane waves in dielectric media

- ✓ Consider the spatial dependence of the **electromagnetic fields**, For **monochromatic waves**, the solution for the spatial dependence, carried by the complex amplitudes **E(r)** and **H(r)**, can be obtained by solving the **Helmholtz equation** $\Longrightarrow$ $\nabla^2\mathbf{E}(\mathbf{r}) + k^2\mathbf{E}(\mathbf{r}) = 0$

$$\nabla^2\mathbf{H}(\mathbf{r}) + k^2\mathbf{H}(\mathbf{r}) = 0$$

- ✓ ***plane wave:*** One of the **easiest** and most **intuitive solutions** for the **Helmholtz equation** also the most frequently used in **optics.**

- ✓ The plane wave is characterised by its *wave vector* **k**, and the mathematical expressions for the **complex amplitudes** are:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-i\mathbf{k}\mathbf{r}} \quad , \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{-i\mathbf{k}\mathbf{r}}$$

- ✓ The magnitudes $\mathbf{E_0}$ and $\mathbf{H_0}$ are now constant vectors

3.5 Monochromatic plane waves in dielectric media

- Each of the Cartesian components of the complex amplitudes **E(r)** and **H(r)** will satisfy the **Helmholtz equation.**

- The modulus of the wave vector **k** is: $k = nk_0 = \left(\omega/c\right)n$

- ω is the angular frequency of the EM plane wave and n is the **refractive index** of the medium where the **wave propagates**.

$$\begin{cases} \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-i\mathbf{kr}} \\ \mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{-i\mathbf{kr}} \end{cases} \Rightarrow \begin{cases} \mathbf{k} \times \mathbf{E}_0 = \omega\mu_0 \mathbf{H}_0 \\ \mathbf{k} \times \mathbf{H}_0 = -\omega\varepsilon \mathbf{E}_0 \end{cases}$$

$$\begin{cases} \nabla \times \mathbf{E} = -i\mu_0\omega\mathbf{H} \\ \nabla \times \mathbf{H} = i\varepsilon\omega\mathbf{E} \end{cases}$$

- These **two formulae**, **valid only** for **plane monochromatic waves**

3.5 Monochromatic plane waves in dielectric media

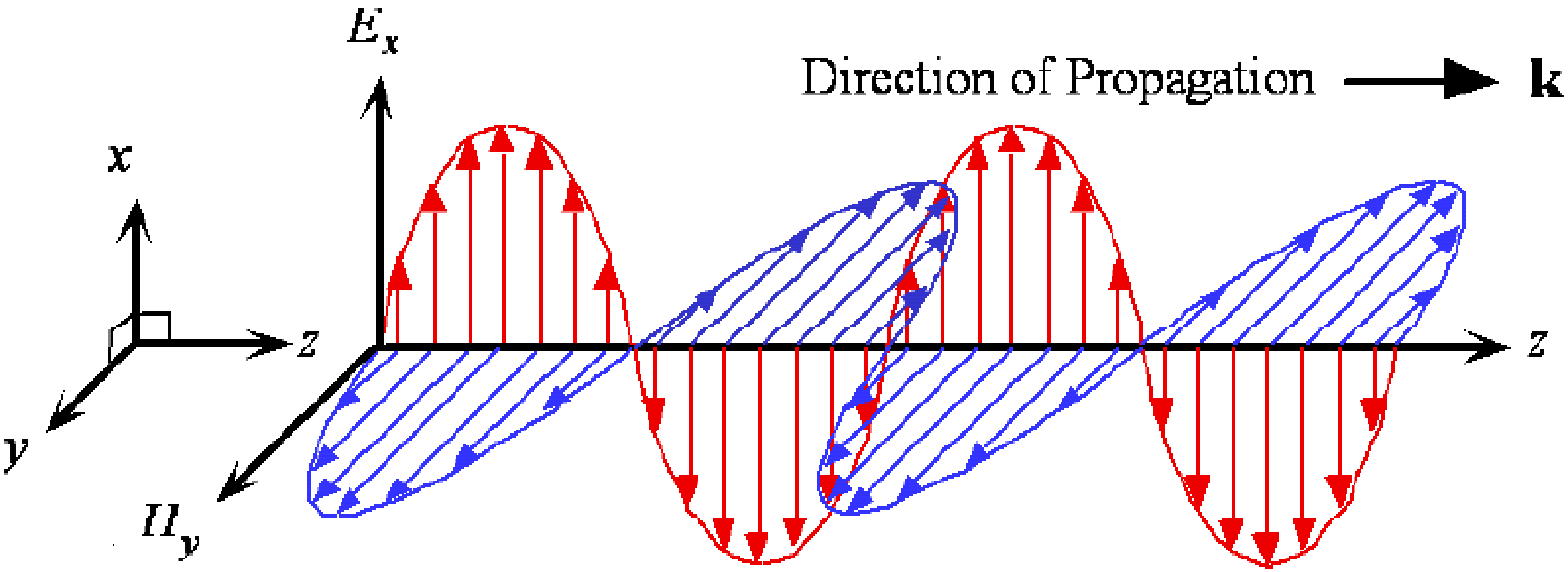
$$\mathbf{k}\times\mathbf{H}_0 = -\omega\varepsilon\mathbf{E}_0 \quad , \quad \mathbf{k}\times\mathbf{E}_0 = \omega\mu_0\mathbf{H}_0$$

- ✓ The **electric field** is **perpendicular** to the **magnetic field** and the wave vector **k**.
- ✓ The **magnetic field** is **perpendicular** to the **electric field** and the wave vector **k**.

- ✓ Therefore, one can conclude that **k**, **E** and **H** are **mutually orthogonal**, and because **E** and **H** *lie* on a **plane normal** to the propagation direction defined by **k**, such **wave** in called a ***transverse EM wave*** (**TEM**).
- ✓ The fact that these three vectors are perpendicular implies

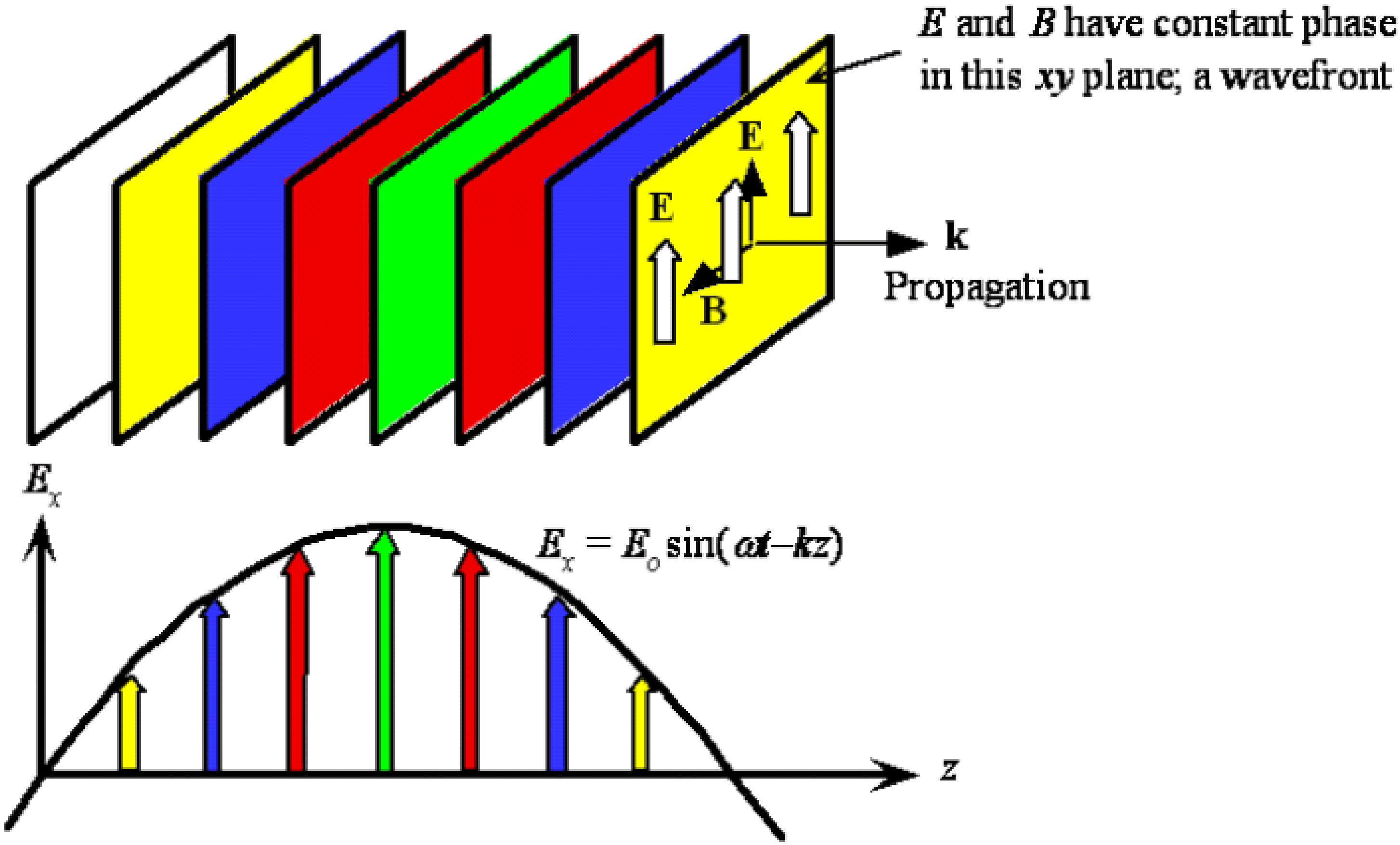
$$\mathbf{H}_0 = \left(\frac{\omega\varepsilon}{k}\right)\mathbf{E}_0 \quad , \quad \left(\frac{k}{\omega\mu_0}\right)\mathbf{E}_0 = \mathbf{H}_0 \quad \Rightarrow \quad k^2 = \omega^2\varepsilon\mu_0$$

The wave nature of light



An electromagnetic wave is a travelling wave which has time varying electric and magnetic fields which are perpendicular to each other and the direction of propagation, z.

The wave nature of light



A plane EM wave travelling along z, has the same E_x (or B_y) at any point in a given ***xy*** plane. All electric field vectors in a given ***xy*** plane are therefore in phase. The ***xy*** planes are of infinite extent in the ***x*** and ***y*** directions.

3.5 Monochromatic plane waves in dielectric media

- When dealing with a monochromatic plane EM wave it is useful to characterise it by its radiation *wavelength* λ, defined as the distance between the two nearest points with equal phase of vibration, measured along the propagation direction. The wavelength is therefore expressed by:

$$\lambda = vT = v/\upsilon = \frac{2\pi}{k} = \frac{2\pi}{nk_0} = \frac{\lambda_0}{n}$$

- λ_0 represents the wavelength of the EM wave in free space, given by:

$$\lambda_0 = cT = c/\upsilon = \frac{2\pi}{k_0}$$

3.5 Monochromatic plane waves in dielectric media

- ✓ It is worth remarking that when an **EM wave** passes from one medium to **another** its frequency remains **unchanged,** but as its phase velocity is **modified** due to its **dependence** on the **refractive index**, the **wavelength** associated with the EM wave should also **change**. Therefore, when the wavelength of an EM wave is given, it is usually referred to the wavelength of that radiation propagating through free space.

$$c = \frac{c_o}{n}, \qquad \lambda = \frac{\lambda_o}{n}, \qquad k = nk_o$$

$$\lambda = 2\pi/k$$

*دسته بندی مدها (قطبش): انتشار در جهت Z فرض شده است.

TE:

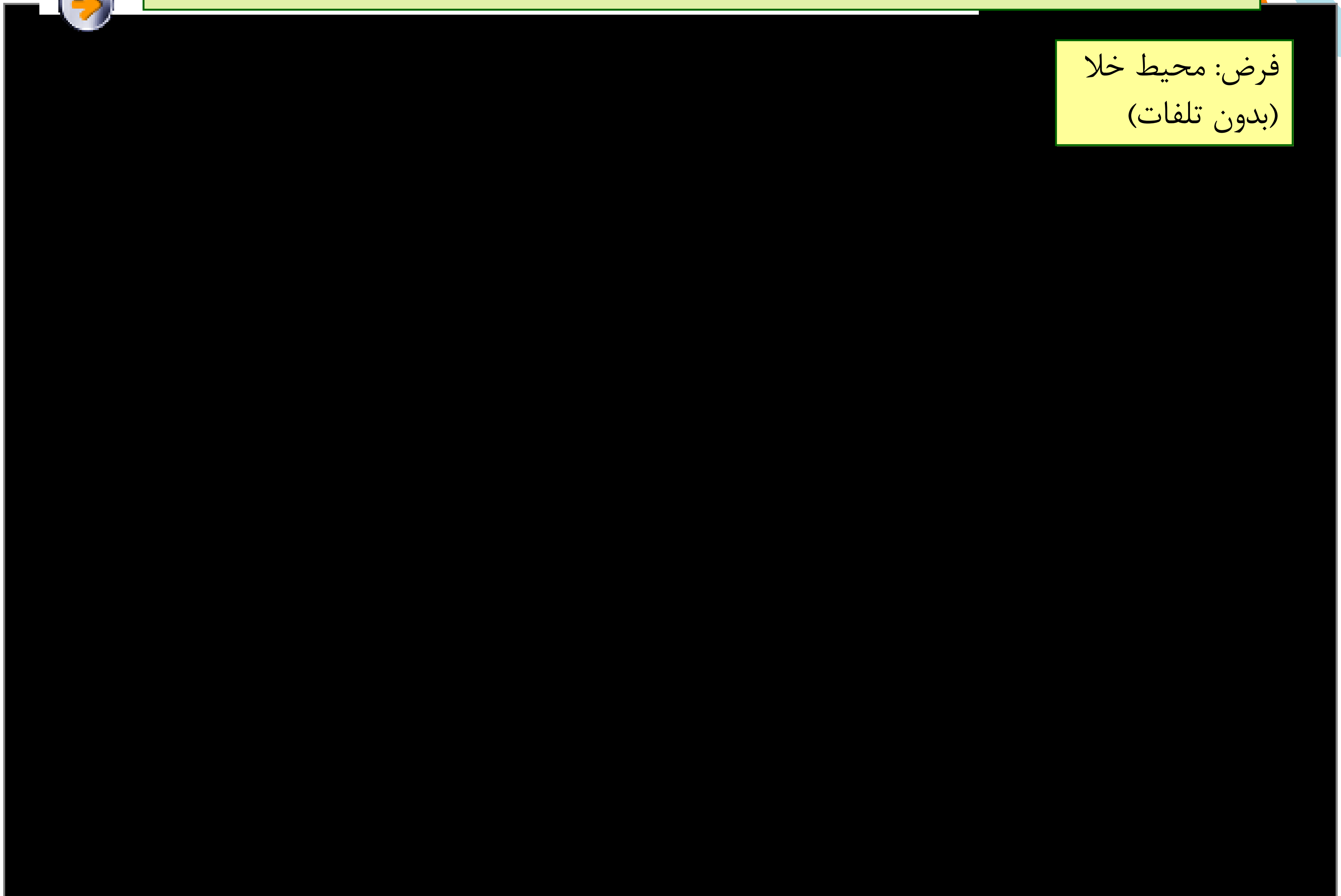
TM:

TEM:

❋حل معادله Helm Holtz (تک رنگ برای امواج با پلاریزاسیون صفحه ای(plane wave):

فرض: محیط خلا

(بدون تلفات)



یعنی **H** فقط در جهت محور **y** مقدار دارد.

امپدانس ذاتی

5. Light propagation in absorbing media

❖

EM radiation dissipated

❖ amplitude plane EM **decreases** exponentially **absorbing**

❖ dielectric permittivity real number complex ε_c

❖ $D = \varepsilon_c E$ will not phase

❖

❖ *n* ***real refractive index*** **κ** ***absorption index***

Proof (1)

Proof (2)

ε [redacted] $\equiv \frac{\varepsilon}{\ }$ [redacted], ε [redacted] $\equiv \frac{\varepsilon_c}{\ }$ [redacted]

[redacted]

Proof(3)

یک محیط بسته به **فرکانس** می تواند در بعضی از فرکانس ها **عایق خوب** و در بعضی دیگر از فرکانس ها **هادی خوبی** باشد.

Good Insulator

Loss angle

Good Conductor

5. Light propagation in absorbing media

$$k_{\circ} = \frac{2\pi}{\lambda_{\circ}} \cdot \frac{\upsilon}{\upsilon} = \frac{\omega}{c}$$

$$k = \frac{2\pi\upsilon}{\lambda\upsilon} = \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda_0} \cdot \frac{\lambda_0}{\lambda} = K_0 \bar{n}$$

$$n_c = \sqrt{\frac{\varepsilon_c}{\varepsilon_0}} = n - i\kappa$$

$$\varepsilon_r \equiv \frac{\varepsilon}{\varepsilon_0}, \varepsilon_{cr} \equiv \frac{\varepsilon_c}{\varepsilon_0}$$

5. Light propagation in absorbing media

- The *complex wavevector:*

$$k_c \equiv k - ia \qquad k_c^2 \equiv \omega^2 \varepsilon_c \mu = n_c^2 k_0^2 \quad \Rightarrow \quad \begin{cases} \mathbf{k}^2 - \mathbf{a}^2 = k_0^2\left(n^2 - \kappa^2\right) \\ \mathbf{ka} = k_0^2 n\kappa \end{cases}$$

- **k** represents the **real wavevector**, and **a** is called the **attenuation** vector.

- The **electric field** for a **plane monochromatic wave** in **absorbing** medium

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}_0 e^{i(\omega t - \mathbf{k}_c \mathbf{r})}\right] = \mathrm{Re}\left[\mathbf{E}_0 e^{-\mathbf{ar}} e^{i(\omega t - \mathbf{kr})}\right]$$

- The planes of constant amplitude will be determined by the condition **ar** = constant. and therefore they will be planes **perpendicular** to the attenuation vector **a**.
- The planes of equal phase will be defined by the condition of **kr** = constant, and thus the phase front will be **planes perpendicular** to the real wavevector **k.**

5. Light propagation in absorbing media

❖ In general, these two planes will not be coincident, and in this case the EM wave is said to be an inhomogeneous wave.

❖ In absorbing media, the vectors κ and a are parallel, and such a wave is called a homogeneous wave.

❖ The vectors k_c, k and a are related to the optical constant of the medium

$$k = nk_0, \qquad a = \kappa k_0 \qquad k_c \equiv (n - i\kappa)k_0$$

❖ The electric field :

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}_0 e^{i(\omega t - n_c \mathbf{k}_0 \mathbf{r})}\right] = \mathrm{Re}\left[\mathbf{E}_0 e^{-\kappa \mathbf{k}_0 \mathbf{r}} e^{i(\omega t - n_c \mathbf{k}_0 \mathbf{r})}\right]$$

❖ One important aspect concerning light propagation in absorbing media is the intensity variation **suffered** by the **wave as propagates**.

❋حل معادله Helm Holtz برای امواج پلاریزاسیون صفحه ای با تلفات:

Lossy media: $\sigma \neq 0$

$$\nabla^2 E + K_c^2 E = 0 \qquad \gamma = ik_c = i\omega\sqrt{\mu\varepsilon_c}$$

$$\gamma = \alpha + i\beta = i\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{i\omega\varepsilon}\right)^{1/2} = i\omega\sqrt{\mu'\varepsilon}\left(1 - i\frac{\varepsilon''}{\varepsilon'}\right)^{1/2}$$

$$\nabla^2 E - \gamma^2 E = 0$$

$$E = a_x E_x = a_x E.e^{-\gamma.z} \Rightarrow E = \hat{a}_x E_0 e^{-\alpha.z} e^{-i\beta.z}$$

α : *attenuation factor*

β : *propagation* constant (or *phase cons.*)

1. Low less media

حالت اول: اگر محیط با تلفات کم باشد: (نیمه هادی ها)

$$\varepsilon'' \langle\langle \varepsilon' \qquad \frac{\sigma}{\omega\varepsilon} \langle\langle 1$$

$$\gamma = \alpha + i\beta \cong i\omega\sqrt{\mu\varepsilon'}\left[1 - i\frac{\varepsilon''}{2\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

$$\alpha \cong \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} \qquad m^{-1}$$

$$\beta \cong \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \qquad \frac{rad}{m}$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu'}{\varepsilon'}}\left[1 - i\frac{\varepsilon''}{\varepsilon'}\right]^{1/2} \cong \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + i\frac{\varepsilon''}{2\varepsilon'}\right)$$

$\frac{\sigma}{\omega\varepsilon} \rangle\rangle 1$ $\varepsilon'' \langle\langle \varepsilon'$ حالت دوم: برای محیط هایی که هادی خوبی باشند

$$\gamma = \alpha + i\beta \cong i\omega\sqrt{\mu\,\varepsilon}\left(1 + \frac{\sigma}{i\omega\varepsilon}\right)^{\frac{1}{2}} \approx \sqrt{i}\sqrt{\mu\,\varepsilon\omega}$$

$$= \frac{1+i}{\sqrt{2}}\sqrt{\mu\omega\sigma} = (1+i)\sqrt{\pi.f.\sigma.\mu} \Rightarrow$$

$$\sqrt{i} = (e^{i\pi/2})^{1/2}$$

$$\Rightarrow \frac{1+i}{\sqrt{2}} = \sqrt{i}$$

$$\alpha = \beta = \sqrt{\pi.f.\sigma.\mu}$$

امپدانس ذاتی یا رابطه بین **H, E**

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \approx \sqrt{\frac{i\omega\mu}{\sigma}} = (1+i)\sqrt{\frac{\pi.f.\mu}{\sigma}} = (1+i)\frac{\alpha}{\sigma}$$

اختلاف فاز میان **E, H** ،**π/4** است. یعنی موج و انرژی آن، جذب فلز می شود یا به طور ایده آل نور در فلز جذب می شود و منتشر نمیگردد.

Depth of penetration (or skin depth)

عمق نفوذ یا عمق پوستی

$$if : f \uparrow \Rightarrow \alpha \uparrow \Rightarrow loss \uparrow$$

$$\delta \underline{\underline{\Delta}} \frac{1}{\alpha}$$

$$at \quad 3kHz \rightarrow \delta = 0.038mm$$

$$at \quad 10GHZ \rightarrow \delta = 0.66\mu m$$

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi . f . \mu . \varepsilon}} \Rightarrow \beta = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\beta} = 2\sqrt{\frac{\pi}{f . \mu . \varepsilon}},$$

$$\delta = \frac{1}{\alpha = \beta} = \frac{\lambda}{2\pi}$$

5. The intensity of light in absorbing media

- we assume that the propagation is along the z-axis; in this case, the intensity takes the form:

$$I(z) = \frac{1}{2c\mu_0} |\mathbf{E}_0|^2 e^{-2\kappa k_0 z}$$

- $I_0 = \frac{1}{2c\mu_0} |\mathbf{E}_0|^2$: the **intensity** associated with the wave at the plane $z = 0$,

$$I(z) = I_0 e^{-2\kappa k_0 z}$$

- The **intensity** of the wave **decreases** *exponentially* as a function of the **propagation distance**. $I(z) = I_0 e^{-\alpha z}$

- The *absorption coefficient* α, defined as: $\alpha \equiv 2\kappa k_0 = 2\kappa \frac{\omega}{c} \quad (\mathrm{m}^{-1})$

- The **light attenuation** in *decibels* (dB), $1 \text{ dB} \equiv 10 \log \left(I_0 / I \right) = 4.3 \alpha d$

6. Metallic media

- A high electrical conductivity σ (compared with $\varepsilon\omega$)

$$\nabla^2\mathbf{E} = \mu\sigma\frac{\partial\mathbf{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} \quad \Rightarrow \nabla^2\mathbf{E}(\mathbf{r}) + \overbrace{\omega^2\mu\underbrace{(\varepsilon - i\sigma/\omega)}_{\varepsilon_G}}^{k_c^2}\mathbf{E}(\mathbf{r}) = 0$$

- ε_G (clearly, a **complex quantity**) is known as generalised **dielectric permittivity.**

- The **Helmholtz equation** is still **valid.**

- Transparent dielectric medium ($\kappa = 0$),

7. EM Waves at planar Dielectric Interfaces

➢ **Boundary conditions at the interface**

✓Another *important aspect* in the study of **light propagation** is the behaviour of **EM waves passing** from **one medium** to **another**.

✓The **behaviour** of an **EM monochromatic plane wave** travelling through a *homogeneous* medium, incident on a second *homogeneous* medium, separated from the former by a planar interface.

✓The equations that determine the reflection and transmission coefficients can be studied separately in two groups:

1) The **electric field** of the incident EM wave has **only** a **parallel component** with respect to the **incident plane** (the magnetic field being perpendicular to that plane)
2) the electric vector has only the component perpendicular to the incident plane

7. EM Waves at planar Dielectric Interfaces

- The **relations** between the **incident**, **reflected** and **transmitted** waves are obtained by setting the adequate **boundary conditions** for the **fields** at **the planar interface**, which are derived directly from **Maxwell's equations.**

- $$\left.\begin{matrix}\nabla.\mathbf{D}=0\\ \nabla.\mathbf{B}=0\end{matrix}\right\} \Rightarrow \begin{cases}\left(\mathbf{D}^{\text{Normal}}\right)_{\mathbf{Medium1}}=\left(\mathbf{D}^{\text{Normal}}\right)_{\mathbf{Medium2}}\\ \left(\mathbf{B}^{\text{Normal}}\right)_{\mathbf{Medium1}}=\left(\mathbf{B}^{\text{Normal}}\right)_{\mathbf{Medium2}}\end{cases}$$ **at interface**

$$\left.\begin{matrix}\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}\\ \nabla\times\mathbf{H}=\mathbf{J}+\dfrac{\partial\mathbf{D}}{\partial t}\end{matrix}\right\} \Rightarrow \begin{cases}\left(\mathbf{E}^{\text{Tangential}}\right)_{\mathbf{Medium1}}=\left(\mathbf{E}^{\text{Tangential}}\right)_{\mathbf{Medium2}}\\ \left(\mathbf{H}^{\text{Tangential}}\right)_{\mathbf{Medium1}}=\left(\mathbf{H}^{\text{Tangential}}\right)_{\mathbf{Medium2}}\end{cases}$$

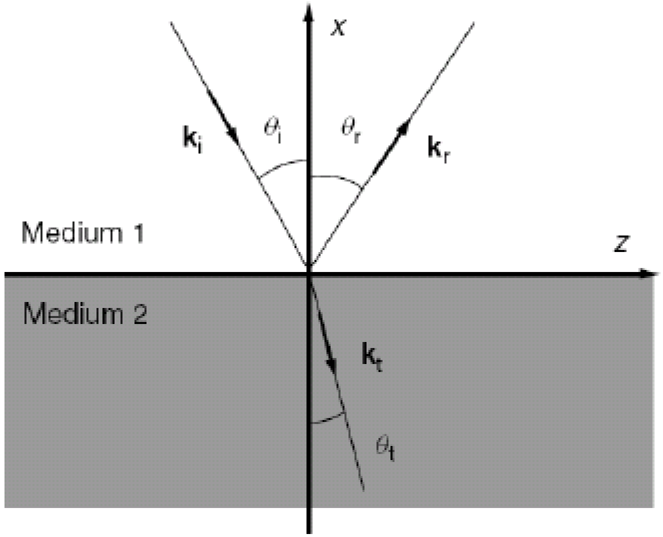
7. EM Waves at planar Dielectric Interfaces

- The dielectric media are characterized by their optical constant (ε_1,μ_1) and (ε_2,μ_2),

$$\mathbf{E}_i(\mathbf{r},t)=\mathbf{E}_i e^{i(\omega_i t-\mathbf{k}_i\mathbf{r})}$$

$$\mathbf{E}_r(\mathbf{r},t)=\mathbf{E}_r e^{i(\omega_r t-\mathbf{k}_r\mathbf{r})}$$

$$\mathbf{E}_t(\mathbf{r},t)=\mathbf{E}_t e^{i(\omega_t t-\mathbf{k}_t\mathbf{r})}$$



- Apply the **condition** of the **continuity** of the **tangential component** of the electric field across the interface

$$\left[\mathbf{E}_i(\mathbf{r},t)+\mathbf{E}_r(\mathbf{r},t)\right]^{\text{Tangential}}=\left[\mathbf{E}_t(\mathbf{r},t)\right]^{\text{Tangential}}$$

$$\left[\mathbf{E}_i e^{i(\omega_i t-\mathbf{k}_i\mathbf{r})}+\mathbf{E}_r e^{i(\omega_r t-\mathbf{k}_r\mathbf{r})}\right]^{\text{Tangential}}=\left[\mathbf{E}_t e^{i(\omega_t t-\mathbf{k}_t\mathbf{r})}\right]^{\text{Tangential}}$$

- As this relation should be valid for any instant of time, it follows that:

$$\omega_i=\omega_r=\omega_t$$

7. EM Waves at planar Dielectric Interfaces

- The **condition** of equal spatial dependence on the exponents **at the interface**

$$k_{iy}y + k_{iz}z = k_{ry}y + k_{rz}z = k_{ty}y + k_{tz}z \qquad (\text{at the interface } \boldsymbol{x} = \mathbf{0})$$

- This result indicates that the **tangential** component of the wavevectors (for the **incident**, **reflected** and **transmitted** **waves**) **must be equal**:

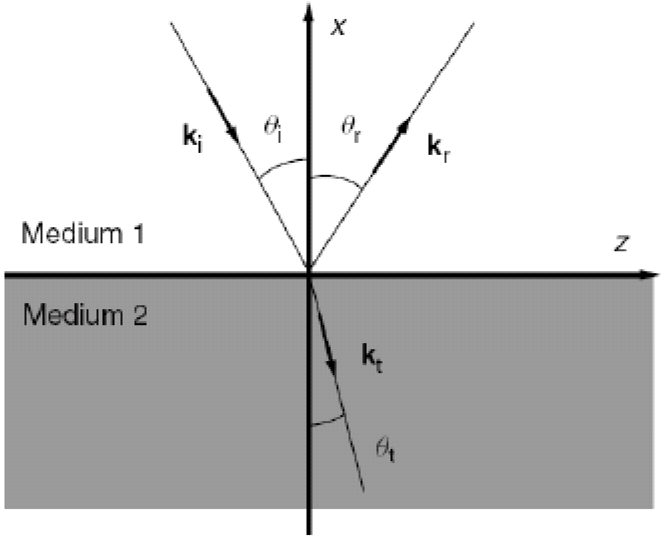
$$[\mathbf{k}_i]^T = [\mathbf{k}_r]^T = [\mathbf{k}_t]^T$$

- In other words, at the **boundary** only the **perpendicular** component of the wavevectors **can change**.
- Thus, the vectors $\mathbf{k}_r$ and $\mathbf{k}_t$ must **lie** in the plane defined by the $\mathbf{k}_i$ vector and the **normal** to the **plane** of the **interface**.

7. EM Waves at **planar** Dielectric Interfaces

- This plane, **perpendicular** to the **plane** that separates both media, is called the ***incident plane***, and all the wavevectors lie on it.

- if we choose the **incident plane** as the x-z plane, in this case the y components of the **wavevectors** are **null:**



$$k_{iz}z = k_{rz}z == k_{tz}z \quad ==>>$$

$$k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$$

$$\left.\begin{aligned} k_i &= \omega(\varepsilon_1\mu_1)^{1/2} = k_r \\ k_t &= \omega(\varepsilon_2\mu_2)^{1/2} \end{aligned}\right\} \Rightarrow \quad \theta_i = \theta_r$$

(*law of reflection*)

$$k_i \sin\theta_i = k_t \sin\theta_t \qquad \textbf{(\textit{Transmission law})}$$

8. Snell's law

- If the two **homogeneous** media are **non-magnetic** *($\mu_1 \approx \mu_2 \approx \mu_0$) and* ***non-absorbing materials*** *(real refractive* indices)

$$\left.\begin{aligned}\left(\varepsilon_1/\varepsilon_0\right)^{1/2} &= \left(\varepsilon_{r1}\right)^{1/2} = n_1 \\ \left(\varepsilon_2/\varepsilon_0\right)^{1/2} &= \left(\varepsilon_{r2}\right)^{1/2} = n_2\end{aligned}\right\} \quad k_i \sin\theta_i = k_t \sin\theta_t \quad \Rightarrow \left(n_1 \sin\theta_i = n_2 \sin\theta_t \right)$$

- ***Snell's law*** is valid for **dielectric materials**.

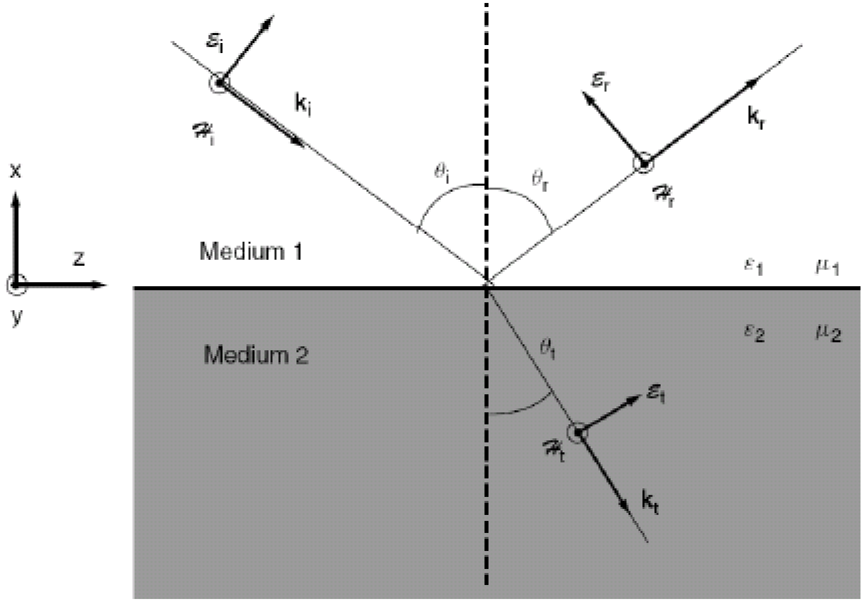
- In the case of **absorbing media**, the equation $k_{iz}z = k_{rz}z == k_{tz}z$ is still **valid**, and is the correct relation to obtain the transmitted wave.

9. Reflection and transmission coefficients

- The **relations** between the **electric field** amplitude for the **incident**, **reflected** and **transmitted waves,**

- **Transverse magnetic incidence (TM incidence or p waves)**
  1) The **electric field vector** associated with the **incident monochromatic plane** wave **lies** on the **incident plane.** **(Parallel Polarization)**

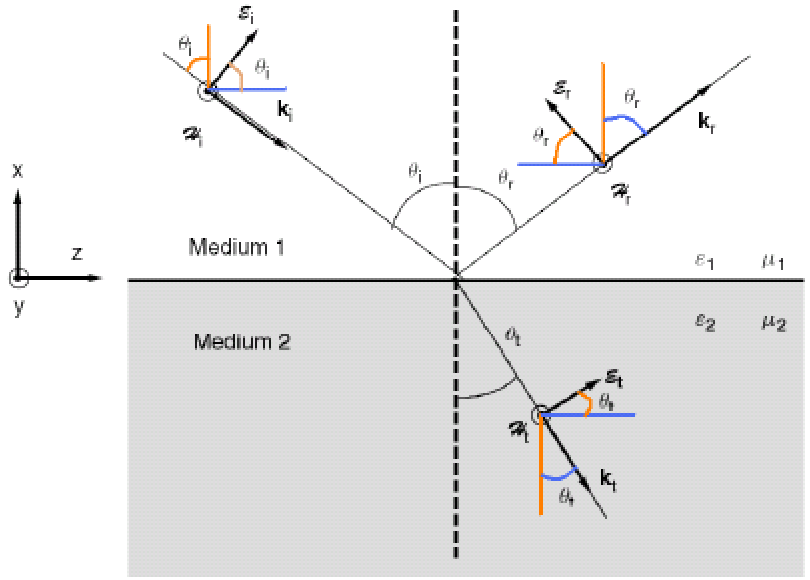
  2) and the **magnetic field** vector is **perpendicular** to **both vectors**



10. Transverse Magnetic incidence (TM incidence or p waves)

$$\mathbf{E}_i \equiv \mathbf{E}_i^{\parallel} \equiv \left[\mathbf{E}_{ix}, 0, \mathbf{E}_{iz}\right]$$

$$\mathbf{H}_i \equiv \mathbf{H}_i^{\perp} \equiv \left[0, \mathbf{H}_{iy}, 0\right]$$



- The symbols || and ⊥ denote vectors parallel and perpendicular to the incident plane, respectively.
- As the **electric field** vector is **paralle** to the incidence plane, the **TM incidence** is also called **parallel incidence**.
- The **condition** of the continuity of the **tangential component** of the **electric field** at the interface:

$$\left(\mathbf{E}^{\text{Tangential}}\right)_{\textbf{Medium1}} = \left(\mathbf{E}^{\text{Tangential}}\right)_{\textbf{Medium 2}} \Rightarrow \mathbf{E}_{iz} + \mathbf{E}_{rz} = \mathbf{E}_{tz}$$

$$\left[\mathbf{E}_i e^{i(\omega_i t - \mathbf{k}_i \mathbf{r})} \cos\theta_i - \mathbf{E}_r e^{i(\omega_r t - \mathbf{k}_r \mathbf{r})} \cos\theta_r\right]_{x=0} = \left[\mathbf{E}_t e^{i(\omega_t t - \mathbf{k}_t \mathbf{r})} \cos\theta_t\right]_{x=0}$$

10. TM incidence (p waves)

- The **temporal** and **spatial dependences** of the **exponentials** are equal (at $x = 0$) $\quad \mathbf{E}_i \cos\theta_i - \mathbf{E}_r \cos\theta_i = \mathbf{E}_t \cos\theta_t$

- The condition of **continuity** of the normal component of the dielectric displacement vector

$$\left(\mathbf{D}^{\text{Normal}}\right)_{\textbf{Medium 1}} = \left(\mathbf{D}^{\text{Normal}}\right)_{\textbf{Medium 2}} \quad \Rightarrow \quad \mathbf{D}_{ix} + \mathbf{D}_{rx} = \mathbf{D}_{tx}$$

$$\varepsilon_1 \mathbf{E}_i \sin\theta_i + \varepsilon_1 \mathbf{E}_r \sin\theta_i = \varepsilon_2 \mathbf{E}_t \sin\theta_t$$

- The relation between the **electric field** amplitudes of the **reflected** and **incident** waves $\quad r_{TM}\left(\Gamma_{\parallel}\right) \equiv \dfrac{\mathbf{E}_r}{\mathbf{E}_i} = \dfrac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}$

- r_{TM} denotes the ***reflection coefficient*** for **parallel polarisation.**
- ***Fresnel equation*** **for the parallel polarization**

10. TM incidence (p waves)

- r_{TM} can also be written in several equivalent forms, one of which is

$$r_{TM}\left(\Gamma_{\parallel}\right) \equiv \frac{\mathbf{E}_r}{\mathbf{E}_i} = \frac{n_2^2\cos\theta_i - n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_i}}{n_2^2\cos\theta_i + n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_i}}$$

- r_{TM} is real when θ_i is smaller than the critical angle, $\sin\phi_c = \frac{n_2}{n_1}$
- r_{TM} can be positive or negative depending on the incidence angle.
- $\Gamma_{\parallel}$ vanishes if $n_1 > n_2$ and if the incidence angle is $\tan\theta_i = \frac{n_2}{n_1}$
- The incidence angle is commonly known as the **polarizing** or **Brewster angle**.

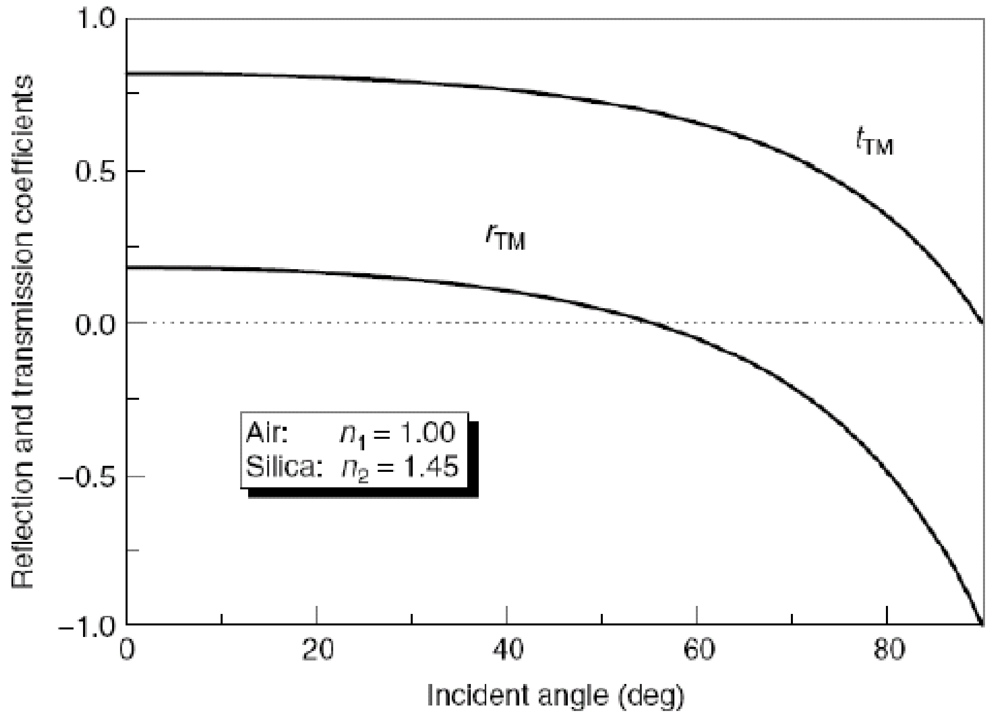
10. TM incidence (p waves)

- The **relation** between the **amplitude** between the transmitted and **incident** waves

$$t_{TM} \equiv \frac{\mathbf{E}_t}{\mathbf{E}_i} = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$

- t_{TM} : the ***transmission coefficient*** for parallel polarisation.

- In general r_{TM} and t_{TM} can be **complex magnitudes.**



10. TM incidence (p waves)

- Although the **reflection** and **transmission coefficients** give us valuable information concerning the **relation** between the electric field amplitudes of the **incident**, **reflected** and **transmitted** waves, in many cases the relevant parameter is the fraction of the incidentene energy that is reflected and transmitted at the interface, defined through **reflectance** and **transmittance**.

- The *reflectance R* is defined as the quotient between the reflected energy in an unit of time over a differential area, and the incident energy per unit of time over the same area at the interface.

- The *transmittance is* defined as the quotient between the transmitted energy per unit of time over a differential area and the incident energy in that unit of time over the same area.

11. Reflectance and transmittance for TM incidence

$$R_{TM} = \left|\frac{\mathbf{E}_r}{\mathbf{E}_i}\right|^2 = |r_{TM}|^2 = \left(\frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}\right)^2$$

$$T_{TM} = \left|\frac{\mathbf{E}_t}{\mathbf{E}_i}\right|^2 = |t_{TM}|^2 = \frac{4n_1n_2\cos\theta_i\cos\theta_t}{(n_2\cos\theta_i + n_1\cos\theta_t)^2}$$

$$\Rightarrow \quad R_{TM} + T_{TM} = 1$$

- From equation $R_{TM} = \left(\frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}\right)^2$ it follows that the reflectance will vanish for the condition $n_2\cos\theta_i = n_1\cos\theta_t$

- By combining R_{TM} with the **Snell's law**, one obtains that the **reflectance** is **zero** for an **incident angle** that fulfils the equation:

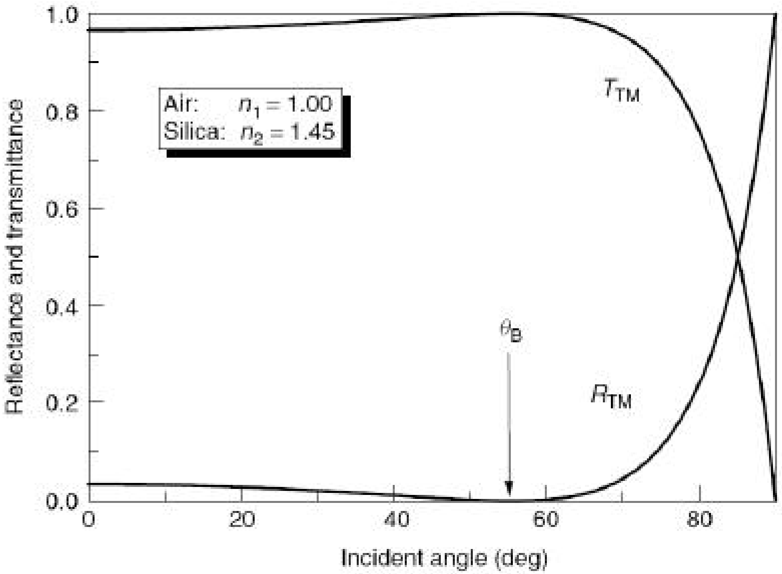
$$\tan\theta_i = n_2 / n_1$$

11. Reflectance and transmittance for TM incidence

- This angle, for which $R_{TM} = 0$, is called Brewster's angle θ_B or the Polarizing angle, because the reflected wave will be linearly polarized for an incident wave with arbitrary polarization state.

- For the particular case of **normal incidence** ($\theta_i = 0$), the formula for the reflectance is simplified to:

$$R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2$$



- Reflectance and transmittance for TM incidence corresponding to the interface air–silica ($n_1 = 1.00$, $n_2 = 1.45$). For an incident angle at $\theta_i = \theta_B$ the reflectance vanishes, corresponding to an angle of 55.4°

12. Brewster's angle or polarization angle (θ_p)

- θ_p is the **angle of incidence** which results in the **reflected wave** having **no electric** field in the **plane of incidence .**
- The electric field oscillations are in the plane perpendicular to the plane of incidence.
- In θ_p, the field in the reflected wave is then always perpendicular to the plane of incidence.
- The reflected wave is then plane polarized.
- This special angle is given by

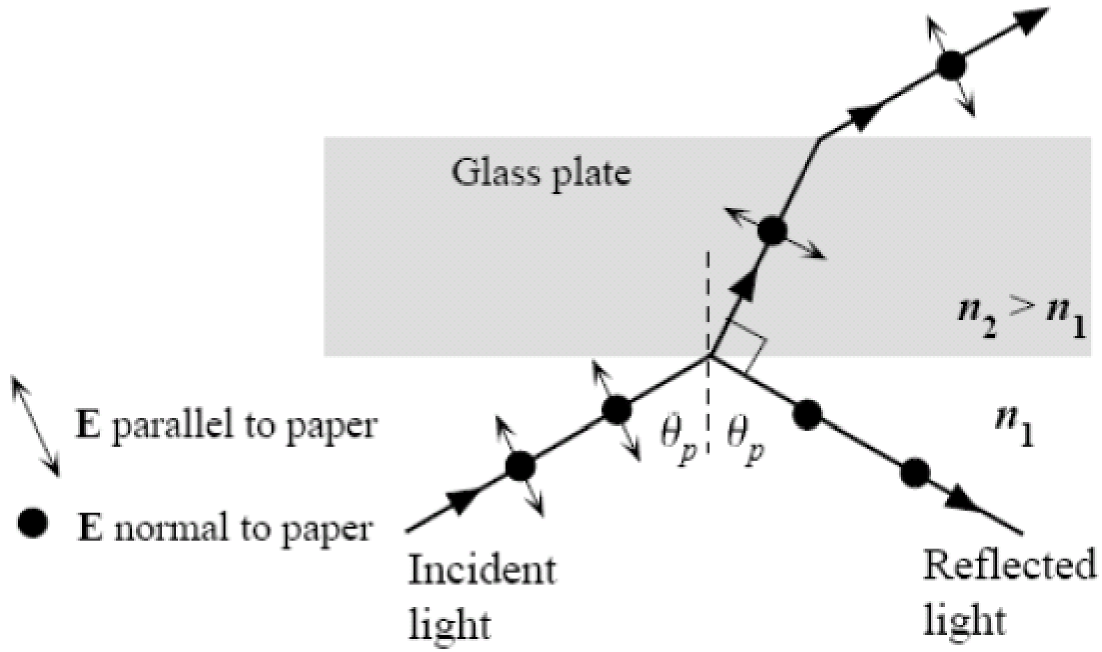
$$\tan\theta_p = \frac{n_2}{n_1}$$

incidence.

12. Brewster's angle or polarization angle (θ_p)

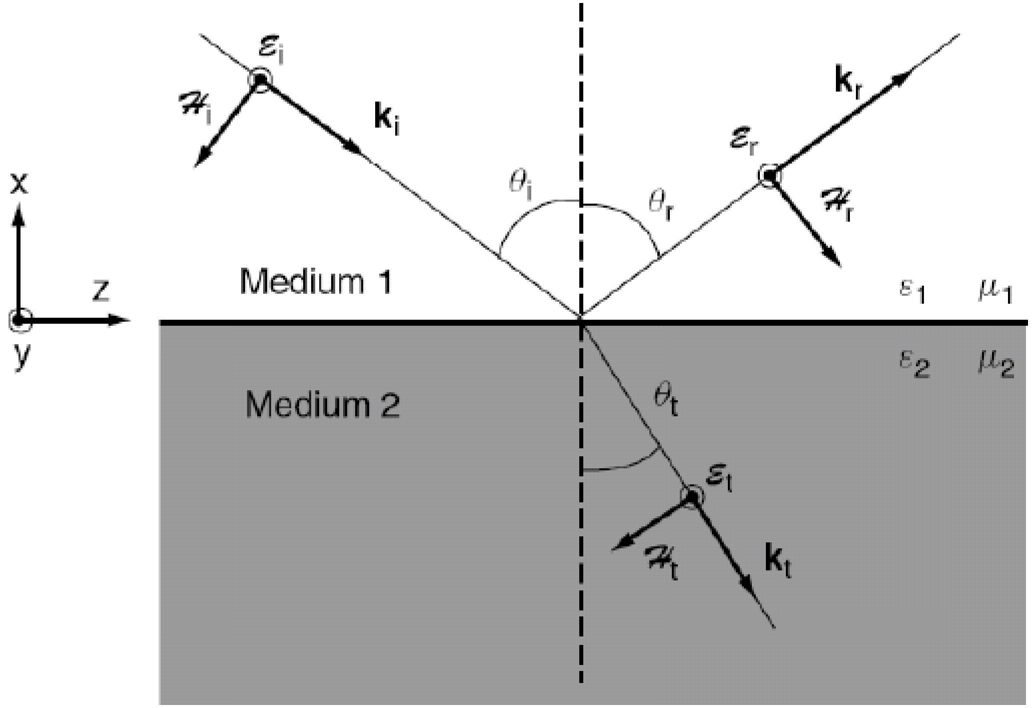
➢ When an **unpolarized light** wave is incident at the **Brewster angle**, the **reflected wave** is polarized with its optical field **normal to** the plane of incidence, that is parallel to the surface of the glass plate.

➢ The angle between the refracted (transmitted) beam and the reflected beam is 90.



13. Transverse Electric incidence (TE incidence or n waves)

➢ The **electric field** vector of the incident wave is **perpendicular** to the **incident plane.**



Reflection and **transmission** corresponding to TE incidence **perpendicular Polarization**.

13. TE incidence (n waves)

➢ The electric and magnetic field vectors associated with the incident wave are:

$$\mathbf{E}_i \equiv \mathbf{E}_i^{\perp} \equiv \left[0, \mathbf{E}_{iy}, 0\right]$$

$$\mathbf{H}_i \equiv \mathbf{H}_i^{\parallel} \equiv \left[\mathbf{H}_{ix}, 0, \mathbf{H}_{iz}\right]$$

➢ The continuity of the tangential component of the electric field across the boundary $\mathbf{E}_{iy} + \mathbf{E}_{ry} = \mathbf{E}_{ty}$

➢ To obtain the reflection and transmission coefficients it is necessary to find a second relation between the electric field amplitudes.

➢ The condition of continuity of the tangential component of the magnetic field vector at the interface: $\mathbf{H}_{iz} + \mathbf{H}_{rz} = \mathbf{H}_{tz}$

13. TE incidence (n waves)

- by relating the magnetic field vectors with the electric field vectors by using equation $\mathbf{k} \times \mathbf{E}_0 = \omega\mu_0\mathbf{H}_0$

- After straightforward calculations, the boundary condition $\mathbf{H}_{iz} + \mathbf{H}_{rz} = \mathbf{H}_{tz}$ becomes: $k_{ix}\left(\mathbf{E}_{iy} - \mathbf{E}_{ry}\right) = k_{tx}\mathbf{E}_{ty}$

- The reflection and transmission coefficients for TE incidence are obtained as a function of the wavevectors:

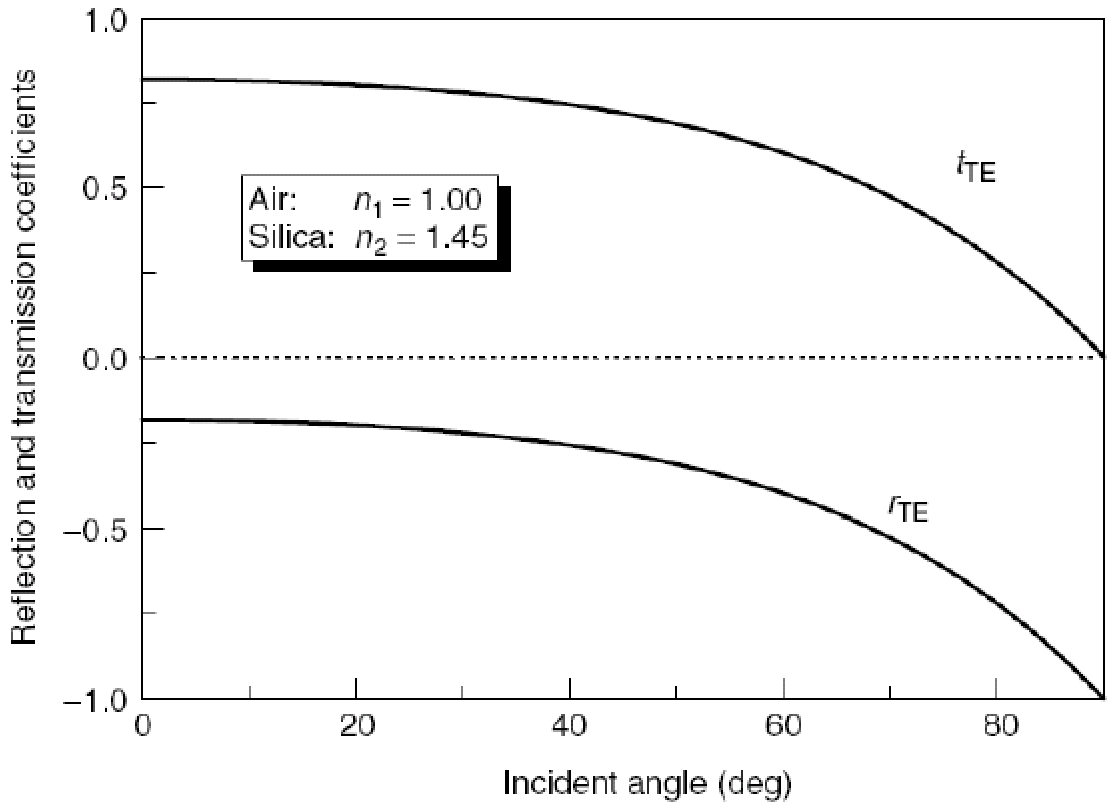
$$r_{TE} \equiv \frac{E_r}{E_i} = \frac{k_{ix} - k_{tx}}{k_{ix} + k_{tx}} \quad , \quad t_{TE} \equiv \frac{E_t}{E_i} = \frac{2k_{ix}}{k_{ix} + k_{tx}}$$

- These coefficients can be expressed in a more convenient form as a function of the incident and refracted angles and the refractive indices of the two media by using Snell's law:

$$r_{TE} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} \quad , \quad t_{TE} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

13. TE incidence (n waves)

➢ The **transmission coefficient** is positive, indicating that the direction of the electric field vector of the transmitted wave is coincident to that of the incident wave.

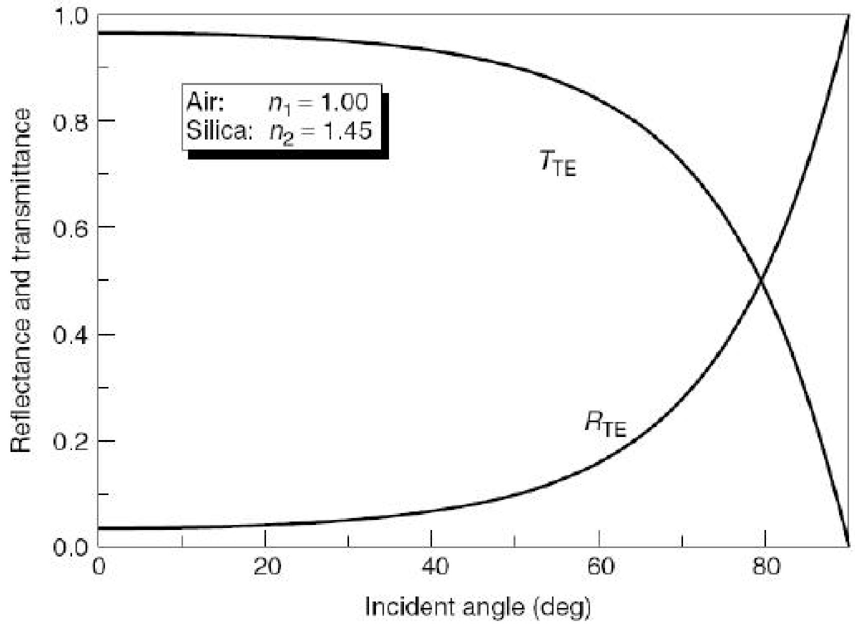


➢ By contrast, the electric field vector associated with the **reflected wave** is **reversed** in respect to that of the incident wave, indicating a phase shift of π in the reflected wave.

14. Reflection and Transmittance for TE incidence

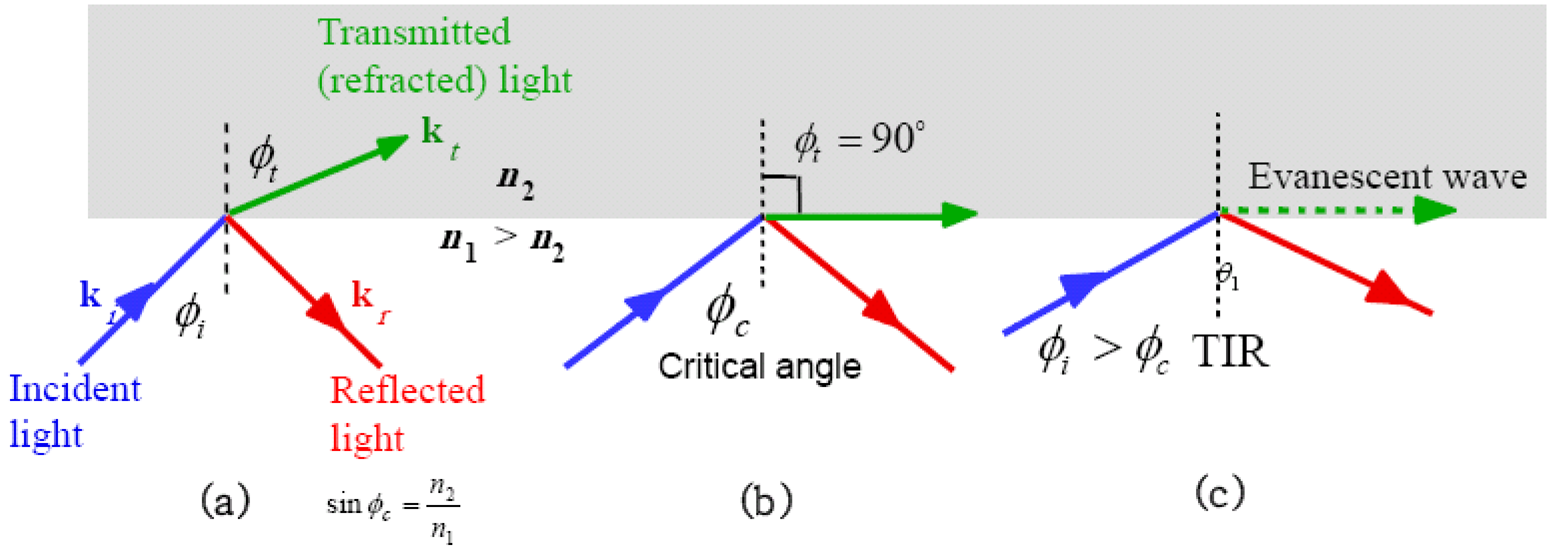
$$R_{TE} = \left|\frac{\mathbf{E}_r}{\mathbf{E}_i}\right|^2 = |r_{TE}|^2 = \left(\frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}\right)^2$$

$$T_{TE} = \left|\frac{\mathbf{E}_t}{\mathbf{E}_i}\right|^2 = |t_{TE}|^2 = \frac{4n_1n_2\cos\theta_i\cos\theta_t}{(n_1\cos\theta_i + n_2\cos\theta_t)^2}$$



- In TE incidence the reflectance is a monotonous increasing function of the incident angle. Therefore, if a beam of non-polarised light is incident at an angle of θ_B, the interface only will reflect the TE component of such radiation, and thus the reflected wave will be linearly polarised with the electric field vector perpendicular to the incident plane. This is the reason why Brewster's angle is also called the polarising angle, and this phenomenon can be used to design polarisation devices.

15. Total internal reflection, Critical angle



Light wave travelling in a more dense medium strikes a less dense medium. Depending on the incidence angle with respect to ϕ_c, which is determined by the ratio of the refractive indices, the wave may be transmitted (refracted) or reflected. (a) $\phi_i < \phi_c$ (b) $\phi_i = \phi_c$ (c) $\phi_i > \phi_c$ and total internal reflection (TIR).

$$\sin\phi_c = \frac{n_2}{n_1}$$

Example

- n (Water)= 1.33, n (glass) = 1.5.
- For most semiconductors, such as Si, GaAs, and InP, the index of refraction is often in the $3 < n < 4$, depending on the optical wavelength and the material.
- Here we take a nominal value of $n = 3.5$ for a semiconductor.
- Find the **reflectance** at ***normal incidence***, the Brewster angles, and the critical angles for these media at their **interfaces** with **air.**

- $R = 0.02$ for water, $R = 0.04$ for ordinary glass, and R typically falls in the range of 0.3 and 0.32 for a semiconductor.
- $\theta_B \approx 54^\circ$ for water, $\theta_B \approx 56^\circ$ for ordinary glass, and θ_B is typically around 74° for a semiconductor.
- $\theta_c \approx 49^\circ$ for water, $\theta_c \approx 42^\circ$ for ordinary glass, and θ_c is around 17° for a semiconductor.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