



Some estimates for the generalized numerical radius

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Abstract

In this article, we explore further properties of the generalized numerical radius. More precisely, we show several inequalities for the generalized numerical radius $w_N(\cdot)$, with some emphasize on p -Schatten numerical radius $w_p(\cdot)$ ($p \geq 1$) of Hilbert space operators.

Keywords Cartesian decomposition · Generalized numerical radius · Schatten p -norm

Mathematics Subject Classification 47A12 · 47A30 · 47A63 · 47B33

1 Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, with identity I . When $\mathcal{H}$ is finite dimensional of dimension n , we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices. For an operator $T \in \mathbb{B}(\mathcal{H})$, we write $T \geq 0$ if T is positive in the sense that $\langle Tx, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$, and $T > 0$ if T is positive and invertible. The notation $\mathcal{M}_n^+(\mathbb{C})$ is used to denote the cone of all positive semidefinite matrices in $\mathcal{M}_n(\mathbb{C})$.

Given $T \in \mathbb{B}(\mathcal{H})$, the usual operator norm and the numerical radius of T are defined respectively as follows:

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\} \text{ and } w(T) = \sup\{|\langle Tx, x \rangle| : \|x\| = 1\}.$$

When talking about the numerical radius, it is necessary to recall the definition of the numerical range $W(T)$ defined for $T \in \mathbb{B}(\mathcal{H})$ by

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$$W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$, where one has the equivalence $\max\{r(T), \frac{1}{2}\|T\|\} \leq w(T) \leq \|T\|$, which $r(T)$ is the spectral radius of $T \in \mathbb{B}(\mathcal{H})$, [14].

Computations of the numerical radius have found some applications in understanding the geometry of the numerical range and in finding bounds on the zeros of complex polynomials, for example. However, it is not easy in general to compute the exact value of $w(T)$ for an arbitrary $T \in \mathbb{B}(\mathcal{H})$. This is why finding upper and lower bounds of $w(T)$ has received a considerable attention in the literature. We refer the reader to [4, 8–13, 15, 17, 19] for further properties and bounds of the numerical radius.

In 2007, Yamazaki [18] proved that for any $T \in \mathbb{B}(\mathcal{H})$, one has the identity

$$w(T) = \sup_{\theta} \|\Re(e^{i\theta}T)\|, \quad (1)$$

where $\Re(\cdot)$ stands for the real part, defined for an operator T by $\Re(T) = \frac{T+T^*}{2}$, in which T^* is the adjoint operator of T .

Very recently, (1) has been utilized in [2] to present a generalization of the numerical radius. Let $N(\cdot)$ be an algebra norm on $\mathbb{B}(\mathcal{H})$. We say that $N(\cdot)$ is self adjoint if $N(T) = N(T^*)$ and weakly unitarily invariant if $N(U^*TU) = N(T)$, for every $T \in \mathbb{B}(\mathcal{H})$ and every unitary $U \in \mathbb{B}(\mathcal{H})$.

Then the generalized numerical radius of $T \in \mathbb{B}(\mathcal{H})$ is defined as follows:

$$w_N(T) = \sup_{\theta \in \mathbb{R}} N(\Re(e^{i\theta}T)),$$

in which $N(\cdot)$ is an algebra norm on $\mathbb{B}(\mathcal{H})$. It is clear how (1) motivates the definition of $w_N(\cdot)$. By the properties of $N(\cdot)$, and the supremum, we see that $w_N(\cdot)$ is a norm.

It was shown in [2] that

$$\frac{1}{2}N(T) \leq w_N(T) \leq N(T) \quad (2)$$

for any $T \in \mathbb{B}(\mathcal{H})$, provided that $N(\cdot)$ is self adjoint. In the same reference, it was shown that if T is self adjoint, then $w_N(T) = N(T)$.

Among the most interesting norms on $\mathbb{B}(\mathcal{H})$ is the so called Hilbert-Schmidt norm $\|\cdot\|_2$. For $T \in \mathbb{B}(\mathcal{H})$, we say that T belongs to the class $HS(\mathcal{H})$ of Hilbert-Schmidt operators in $\mathbb{B}(\mathcal{H})$ if $\|T\|_2 := (\text{tr}T^*T)^{1/2} < \infty$. Recall that $\|\cdot\|_2$ is unitarily invariant in the sense that $\|UTV\|_2 = \|T\|_2$ for any unitaries $U, V \in \mathbb{B}(\mathcal{H})$ and $T \in \mathbb{B}(\mathcal{H})$.

When $N(\cdot)$ is the Hilbert-Schmidt norm $\|\cdot\|_2$, $w_2(\cdot)$ is defined by

$$w_2(T) = \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta}T)\|_2.$$

It has been shown in [2] that $w_2(T) = \sqrt{\frac{1}{2}\|T\|_2^2 + \frac{1}{2}|\text{tr}(T^2)|}$ and

$$\left\{ r(T), \frac{1}{\sqrt{2}} \|T\|_2 \right\} \leq w_2(T) \leq \|T\|_2. \quad (3)$$

If $\text{tr}(T^2) = 0$, then $w_2(T) = \frac{1}{\sqrt{2}} \|T\|_2$ and if T is normal, then $w_2(T) = \|T\|_2$. Hence, the inequalities in (3) are sharp.

A unitary operator U is said to be cramped if its spectrum $\sigma(U)$ is contained in some open semi-circle (of the unit circle). That is

$$\sigma(U) \subseteq \{e^{i\theta} : \alpha < \theta < \alpha + \pi\}.$$

We will use the notation θ_U to denote the central angle of the smallest sector centered at 0 and containing $\sigma(U)$.

Noting (2) and the fact that $N(\cdot)$ is an algebra norm, we deduce

$$w_N(TX) \leq N(TX) \leq N(T)N(X) \leq 4w_N(T)w_N(X) \quad (4)$$

for any self adjoint algebra norm $N(\cdot)$ and $T, X \in \mathbb{B}(\mathcal{H})$.

Given $T \in \mathbb{B}(\mathcal{H})$, the vector function $T - \lambda I$ is known as the pencil generated by $T \in \mathbb{B}(\mathcal{H})$. For the self adjoint norm $N(\cdot)$, $D_{N,T}$ is defined as follows:

$$D_{N,T} = \inf_{\lambda \in \mathbb{C}} N(T - \lambda I),$$

where there is at least $\lambda_0 \in \mathbb{C}$ such that $\inf_{\lambda \in \mathbb{C}} N(T - \lambda I) = N(T - \lambda_0 I)$; see [6].

The authors in [6], showed a refinement of inequality (4) for any $T, X \in HS(\mathcal{H})$ and for any self adjoint norm $N(\cdot)$ on $\mathbb{B}(\mathcal{H})$ as follow:

$$w_N(TX) \leq (N(T) + D_{N,T})w_N(X) \leq 2N(T)w_N(X) \leq 4w_N(T)w_N(X). \quad (5)$$

As a special case of (5), we have

$$w_2(TX) \leq (\|T\|_2 + D_{\|\cdot\|_2, T})w_2(X) \leq 2\|T\|_2 w_2(X) \leq 4w_2(T)w_2(X). \quad (6)$$

Using the inequalities in (2) for any $T, U \in \mathcal{M}_n(\mathbb{C})$, where U is unitary, we have

$$w_2(UT) \leq w_2(UT) = \|UT\|_2 = \|T\|_2 \leq 2w_2(T). \quad (7)$$

Let $R_{\|\cdot\|, T}$ be the radius of the smallest disc in the complex plane containing $\sigma(T)$. It is known that for any normal operator T , we have $D_{\|\cdot\|, T} = R_{\|\cdot\|, T}$; (see [16]). Further, if T is a self adjoint operator, we have $D_{\|\cdot\|, T} = \frac{1}{2}(\lambda_2 - \lambda_1)$, where $\lambda_1 = \min_{\lambda \in \sigma(T)} \lambda$ and $\lambda_2 = \max_{\lambda \in \sigma(T)} \lambda$.

In this paper, we obtain several inequalities about the Hilbert-Schmidt numerical radius for the product of operators. For instance, we present refinements of (6), and find inequality for (7), when U is cramped. Moreover, we explore some inequalities for the p -Schatten numerical radius.

2 Main results

We begin with the following lemmas, which we need in our discussion.

Lemma 1 [6] For any $T, X \in \mathbb{B}(\mathcal{H})$ the following statements hold:

- (i) $w_N(T) = \sup_{\theta \in \mathbb{R}} N(\Im(e^{i\theta}T))$, where $\Im$ denotes the imaginary part;
- (ii) $w_N(T) \leq \frac{1}{2}(N(T) + N(T^*))$;
- (iii) $w_N(TX \pm XT^*) \leq (N(T) + N(T^*))w_N(X)$.

Lemma 2 [7] Let $A \in HS(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{H})$. Then

$$\max\{\|AB\|_2, \|BA\|_2\} \leq \|A\|_2 \|B\|.$$

This lemma is utilized in the following results for $\mathcal{M}_n(\mathbb{C})$. In the following, we show several generalizations of known inequalities for which we used some techniques from [1].

Proposition 1 If $T, X \in HS(\mathcal{H})$ such that T is positive, then

$$w_2(TX) \leq \frac{1}{2}\|T\|_2(\|X\|_2 + w_2(X)). \quad (8)$$

Proof We have

$$\begin{aligned} w_2(TX) &= w_2\left(\left(T - \frac{1}{2}\|T\|_2 I\right)X + \frac{1}{2}\|T\|_2 X\right) \\ &\leq w_2\left(\left(T - \frac{1}{2}\|T\|_2 I\right)X\right) + \frac{1}{2}\|T\|_2 w_2(X) \\ &\leq \left\|\left(T - \frac{1}{2}\|T\|_2 I\right)\right\|_2 \|X\|_2 + \frac{1}{2}\|T\|_2 w_2(X) \\ &= \frac{1}{2}\|T\|_2 \|X\|_2 + \frac{1}{2}\|T\|_2 w_2(X), \end{aligned}$$

where the last identity follows from the fact that $\left\|T - \frac{1}{2}\|T\|_2 I\right\|_2 = \frac{1}{2}\|T\|_2$, since T is positive. $\square$

Theorem 2 Suppose $T, U \in \mathcal{M}_n(\mathbb{C})$ such that U is cramped. Then

$$\frac{1}{c(\theta_U)} w_2(T) \leq w_2(UT) \leq c(\theta_U) w_2(T), \quad (9)$$

where $c(\theta_U) = \sqrt{n} + \sqrt{\frac{n}{2}\sqrt{1 - \cos \theta_U}}$.

Proof We consider the smallest open semi-circle $\{e^{i\theta} : \theta_1 \leq \theta \leq \theta_2, \theta_1, \theta_2 \in \mathbb{R}\}$ containing $\sigma(U)$. Then $\theta_U = \theta_2 - \theta_1$. From Lemma 2 for any $n \times n$ unitary operator matrix U and the identity matrix I , we see that

$$\begin{aligned}
D_{\|\cdot\|_2, U} &= \inf_{\lambda \in \mathbb{C}} \|U - \lambda I\|_2 \\
&= \inf_{\lambda \in \mathbb{C}} \|I(U - \lambda I)\|_2 \\
&\leq \inf_{\lambda \in \mathbb{C}} \|U - \lambda I\| \|I\|_2 \\
&= \frac{1}{2} |e^{i\theta_1} - e^{i\theta_2}| \sqrt{n} \quad (\text{since } D_{\|\cdot\|, U} = R_{\|\cdot\|, U}) \\
&= \frac{1}{\sqrt{2}} \sqrt{1 - \cos \theta_U} \sqrt{n}.
\end{aligned}$$

Hence by inequality (5), we have

$$w_2(UT) \leq (\|U\|_2 + D_{\|\cdot\|_2, U}) w_2(T) = c(\theta_U) w_2(T). \quad (10)$$

Moreover, the first inequality follows from the fact $w_2(T) = w_2(U^*UT) \leq c(\theta_{U^*}) w_2(UT)$ and inequality (10). This completes the proof. $\square$

Remark 1 If $0 \leq \theta \leq \pi$, then $\sqrt{n} < c(\theta) < 2\sqrt{n}$. Therefore

$$w_2(UT) \leq c(\theta_U) w_2(T) \leq 2\sqrt{n} w_2(T).$$

Theorem 3 Suppose $T \in \mathcal{M}_n(\mathbb{C})$ and $0 \notin W(T)$. Then for any matrix A ,

$$w_2(TA) \leq \frac{1}{2} \|T\|_2 w_2(A) \left(\sqrt{2} + c(\theta_T) \right), \quad (11)$$

where $c(\theta_T) = \sqrt{n} + \sqrt{\frac{n}{2}} \sqrt{1 - \cos \theta_T}$.

Proof Since $W(T)$ is convex and $0 \notin \overline{W(T)}$, there is a sector $\{re^{i\theta} : r > 0, \theta_1 \leq \theta \leq \theta_2, \theta_2 - \theta_1 < \pi\}$, which contains $W(T)$. Since T is invertible, T has the polar decomposition, $T = U|T|$, where U is unitary. Moreover, U is cramped [14, p.35]. According to the definition of θ_U , we see that $\theta_U \leq \theta_T$. Since $w_2(TA) = w_2(U|T|A) = w_2(|T|AU)$, we have

$$\begin{aligned}
w_2(TA) &= w_2(|T|AU) \leq \frac{1}{2} \|T\|_2 (\|AU\|_2 + w_2(AU)) \quad (\text{by (8)}) \\
&\leq \frac{1}{2} \|T\|_2 (\|A\|_2 + w_2(UA)) \\
&\quad (\text{since } \|\cdot\| \text{ is a unitarily invariant norm}) \\
&\leq \frac{1}{2} \|T\|_2 (\|A\|_2 + c(\theta_U) w_2(A)) \quad (\text{by Theorem 2}) \\
&\leq \frac{1}{2} \|T\|_2 (\|A\|_2 + c(\theta_T) w_2(A)) \\
&\leq \frac{1}{2} \|T\|_2 w_2(A) \left(\sqrt{2} + c(\theta_T) \right) \quad (\text{by (3)}).
\end{aligned}$$

$\square$

The next lemma is a scalar inequality that we will need to show an inequality for $w_2^2(\cdot)$. The proof of this lemma is straightforward, so we omit it.

Lemma 3 If $a, b \geq 0$, then

$$\frac{a^2 + b^2}{2} = ab + \frac{1}{2}(a - b)^2. \quad (12)$$

Now, we obtain an upper bound for $w_2^2(\cdot)$.

Theorem 4 Let $T \in HS(\mathcal{H})$. Then

$$w_2^2(T) \leq \|T\|_2^2 + |trT^2| - \|T\|_2 \cdot |trT^2|^{1/2}. \quad (13)$$

Proof By putting $a = \|T\|_2$ and $b = |trT^2|^{1/2}$ in (12), we have

$$\begin{aligned} w_2^2(T) &= \frac{\|T\|_2^2 + |trT^2|}{2} \\ &= \|T\|_2 \cdot |trT^2|^{1/2} + \frac{1}{2} \left(\|T\|_2 - |trT^2|^{1/2} \right)^2 \\ &\leq \frac{\|T\|_2^2 + |trT^2|}{2} + \frac{1}{2} \|T\|_2^2 + \frac{1}{2} |trT^2| - \|T\|_2 |trT^2|^{1/2} \\ &= \|T\|_2^2 + |trT^2| - \|T\|_2 \cdot |trT^2|^{1/2}, \end{aligned}$$

which completes the proof. $\square$

Remark 2 If $T \in HS(\mathcal{H})$ such that $tr(T^2) = 0$, then $w_2^2(T) \leq \|T\|_2^2$.

In the following, we obtain a lower bound for $w_N(\cdot)$.

Theorem 5 Let $T \in \mathbb{B}(\mathcal{H})$ and $N(\cdot)$ be a self-adjoint algebra norm. Then

$$w_N(T) \geq \frac{1}{2} \sqrt{|N(|T|^2 + |T^*|^2) - 2w_N(T^2)|}. \quad (14)$$

Proof We have

$$\begin{aligned} w_N(T) &= \sup_{\theta \in \mathbb{R}} N(\Re(e^{i\theta} T)) \\ &= \sup_{\theta \in \mathbb{R}} N\left(\frac{e^{i\theta} T + e^{-i\theta} T^*}{2}\right) \\ &\geq \frac{1}{2} \sup_{\theta \in \mathbb{R}} \sqrt{N(e^{i\theta} T + e^{-i\theta} T^*)^2} \\ &\quad (\text{since } N(\cdot) \text{ is an algebra norm}) \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \sqrt{N(|T|^2 + |T^*|^2 + 2\Re(e^{2i\theta} T^2))} \\ &\geq \frac{1}{2} \sqrt{|N(|T|^2 + |T^*|^2) - 2 \sup_{\theta \in \mathbb{R}} N(\Re(e^{2i\theta} T^2))|} \\ &= \frac{1}{2} \sqrt{|N(|T|^2 + |T^*|^2) - 2w_N(T^2)|}. \end{aligned}$$

This proves the inequality. $\square$

3 Further inequalities for w_p

In this section, we give some lower and upper bounds for the p -Schatten numerical radius $w_p(\cdot)$. The notation $c_p(s, t) := (2^{p/2} - 2) \min(|s|^p, |t|^p)$ for $0 < p < \infty$ and $s, t \in \mathbb{C}$ will be used in the sequel. Also, we remind the reader of the notation $s_j(T)$ which means the j -th singular value of the matrix T , when these singular values are arranged decreasingly, counting multiplicities. So, $s_1(T) \geq s_2(T) \geq \cdots \geq s_n(T)$.

Theorem 6 *Let $T \in \mathcal{M}_n(\mathbb{C})$.*

(i) *If $1 \leq p \leq 2$, then*

$$\begin{aligned} & 2^{\frac{3p}{2}-2} \|T\|_p^p - n \inf_{\theta \in \mathbb{R}} c_p(s_n(\Re(e^{i\theta}T)), s_n(\Im(e^{i\theta}T))) \\ & \leq 2w_p^p(T) \\ & \leq 2^{3-3p/2} \|T\|_p^p + n(2^{2-p} - 2^{3-3p/2}) |s_n(e^{i\theta}T)|^p. \end{aligned} \quad (15)$$

(ii) *If $2 < p < \infty$, then*

$$\begin{aligned} & 2^{3-\frac{3p}{2}} \|T\|_p^p + n(2 - 2^{2-\frac{p}{2}}) \sup_{\theta \in \mathbb{R}} |s_n(e^{i\theta}T)|^p \\ & \leq 2w_p^p(T) \\ & \leq 2^{\frac{p}{2}-1} \|T\|_p^p - n \inf_{\theta \in \mathbb{R}} c_p(s_n(\Re(e^{i\theta}T)), s_n(\Im(e^{i\theta}T))). \end{aligned} \quad (16)$$

Proof We prove (15). By [3, Theorem 4.1], we have

$$\begin{aligned} & 2^{\frac{3p}{2}-2} (\|A\|_p^p + \|B\|_p^p) - nc_p(s_n(A+B), s_n(A-B)) \\ & \leq \|A+B\|_p^p + \|A-B\|_p^p \\ & \leq 2^{2-\frac{p}{2}} (\|A\|_p^p + \|B\|_p^p) + n2^{2-\frac{p}{2}} c_p(s_n(A), s_n(B)), \end{aligned}$$

for $1 \leq p \leq 2$ and $A, B \in \mathcal{M}_n(\mathbb{C})$.

Let $A = e^{i\theta}T/2$ and $B = e^{-i\theta}T^*/2$. Then $A+B = \Re(e^{i\theta}T)$ and $A-B = i\Im(e^{i\theta}T)$ for any $\theta \in \mathbb{R}$. Applying the above inequalities implies

$$\begin{aligned} & 2^{\frac{3p}{2}-2} (\|e^{i\theta}T/2\|_p^p + \|e^{-i\theta}T^*/2\|_p^p) - nc_p(s_n(\Re(e^{i\theta}T)), s_n(i\Im(e^{i\theta}T))) \\ & \leq \|\Re(e^{i\theta}T)\|_p^p + \|\Im(e^{i\theta}T)\|_p^p \\ & \leq 2^{2-\frac{p}{2}} (\|e^{i\theta}T/2\|_p^p + \|e^{-i\theta}T^*/2\|_p^p) + n2^{2-\frac{p}{2}} c_p(s_n(e^{i\theta}T/2), s_n(e^{-i\theta}T^*/2)), \end{aligned}$$

which is equivalent to saying

$$\begin{aligned}
& 2^{\frac{p}{2}-2} \|T\|_p^p - nc_p(s_n(\Re(e^{i\theta}T)), s_n(\Im(e^{i\theta}T))) \\
& \leq \|\Re(e^{i\theta}T)\|_p^p + \|\Im(e^{i\theta}T)\|_p^p \\
& \leq 2^{3-3p/2} \|T\|_p^p + n(2^{2-p} - 2^{3-3p/2}) |s_n(e^{i\theta}T)|^p.
\end{aligned}$$

Now, by taking the supremum over $\theta \in \mathbb{R}$, we get the desired result. $\square$

Remark 3 In Theorem 6 if we take $p = 2$, we get a refinement of inequality (3).

In the next result, we obtain a refinement of the square for inequality (2) for the p -Schatten norms. For this, we need the following lemma [5].

Lemma 4 Let $A, B \in \mathcal{M}_n(\mathbb{C})$. Then

- (i) $\|A\|_p^2 + (p-1)\|B\|_p^2 \geq \frac{1}{2}(\|A+B\|_p^2 + \|A-B\|_p^2)$ for $1 \leq p \leq 2$
- (ii) $\|A\|_p^2 + (p-1)\|B\|_p^2 \geq \frac{1}{2}(\|A+B\|_p^2 + \|A-B\|_p^2)$ for $2 \leq p < \infty$.

Proposition 7 Suppose $T \in \mathcal{M}_n(\mathbb{C})$. Then

- (i) $\frac{p}{4} \|T\|_p^2 \leq w_p^2(T)$ for $1 \leq p \leq 2$.
- (ii) $w_p^2(T) \leq \frac{p}{4} \|T\|_p^2$ for $2 \leq p < \infty$.

Proof The results follow immediately from Lemma 4 by the same method in the proof of Theorem 6. $\square$

Remark 4 We note that the inequalities (i) and (ii) (for $2 \leq p \leq 4$) in Proposition 7 are refinements of the first and the second inequalities in (2), respectively.

We conclude this article by presenting some upper and lower bounds for p^{th} power of $w_p(\cdot)$.

Proposition 8 Let $T \in \mathcal{M}_n(\mathbb{C})$.

- (i) If $1 \leq p \leq 2$, then

$$\frac{1}{4} \left(1 + (p-1)^{p/2}\right) \|T\|_p^p - \frac{n}{2} c_p(s_n(\Re(e^{i\theta}T)), s_n(\Im(e^{i\theta}T))) \leq w_p^p(T).$$

- (ii) If $2 \leq p < \infty$, then

$$\frac{1}{4} \left(1 + (p-1)^{p/2}\right) \|T\|_p^p - \frac{n}{2} c_p(s_n(\Re(e^{i\theta}T)), s_n(\Im(e^{i\theta}T))) \geq w_p^p(T).$$

Proof The proof follows easily from the inequalities which obtained in [3, Theorem 4.3]. $\square$

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