



Some estimations of the Berezin radius and the Berezin norm

Mojtaba Bakherad¹ · Fuad Kittaneh^{2,3}

Received: 21 August 2024 / Accepted: 3 April 2025

© The Author(s), under exclusive licence to Springer-Verlag Italia S.r.l., part of Springer Nature 2025

Abstract

The Berezin range of a bounded operator A acting on a reproducing kernel Hilbert space $\mathcal{H}$ is the set $\mathbf{Ber}(A) := \{\langle A\hat{k}_\tau, \hat{k}_\tau \rangle : \tau \in \Theta\}$, where $\hat{k}_\tau$ is the normalized reproducing kernel for $\mathcal{H}$ at $\tau \in \Theta$. The Berezin radius (number) and the Berezin norm of an operator A are defined by $\mathbf{ber}(A) := \sup_{\tau \in \Theta} |\langle A\hat{k}_\tau, \hat{k}_\tau \rangle|$ and $\|A\|_{\mathbf{ber}} := \sup_{\tau, \mu \in \Theta} |\langle A\hat{k}_\tau, \hat{k}_\mu \rangle|$, respectively. In this paper, we obtain some estimations for the Berezin radius and the Berezin norm. It is shown, among other inequalities, that if $A, B \in \mathcal{L}(\mathcal{H})$ and $1 \leq r \leq 2$, then

$$\begin{aligned} \mathbf{ber}^r(A^*B) &\leq \frac{1}{2^{r\mu}p} \mathbf{ber}^{r(1-\mu)}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{r\mu} \\ &\quad + \frac{1}{2^{r(1-\nu)}q} \mathbf{ber}^{r\nu}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{r(1-\nu)} \\ &\leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}} \end{aligned}$$

for all $\mu, \nu \in [0, 1]$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Keywords Cauchy–Schwarz inequality · Numerical radius · Mean

Mathematics Subject Classification Primary 47A30 · Secondary 47A63 · 15A60

1 Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators defined on $\mathcal{H}$. A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ such that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau \in \Theta$, are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem, there is a unique

✉ Fuad Kittaneh
fkitt@ju.edu.jo

Mojtaba Bakherad
mojtaba.bakherad@gmail.com; bakherad@math.usb.ac.ir

¹ Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Islamic Republic of Iran

² Department of Mathematics, The University of Jordan, Amman, Jordan

³ Department of Mathematics, Korea University, Seoul 02841, South Korea

element $k_\tau \in \mathcal{H}$ such that $f(\tau) = \langle f, k_\tau \rangle$ for all $f \in \mathcal{H}$ and every $\tau \in \Theta$. The function k on $\Theta \times \Theta$ defined by $k(z, \tau) = k_\tau(z)$ is called the reproducing kernel of $\mathcal{H}$, see [1]. It was shown that $k_\tau(z)$ can be represented by

$$k_\tau(z) = \sum_{n=1}^{\infty} \overline{e_n(\tau)} e_n(z)$$

for any orthonormal basis $\{e_n\}_{n \geq 1}$ of $\mathcal{H}$. For example, for the Hardy-Hilbert space $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\{z^n\}_{n \geq 1}$ is an orthonormal basis. Therefore, the reproducing kernel of $\mathcal{H}^2$ is the function $k_\tau(z) = \sum_{n=1}^{\infty} \overline{z^n} z^n = (1 - \bar{\tau}z)^{-1}$, $\tau \in \mathbb{D}$.

Let $\hat{k}_\tau = \frac{k_\tau}{\|k_\tau\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$ (RKHS). For a given bounded linear operator A on $\mathcal{H}$, the Berezin symbol (or Berezin transform) of A is the bounded function $\tilde{A}$ on Θ defined by

$$\tilde{A}(\tau) = \langle A\hat{k}_\tau, \hat{k}_\tau \rangle, \quad \tau \in \Theta.$$

An important property of the Berezin symbol is that for all $A, B \in \mathcal{L}(\mathcal{H})$, if $\tilde{A}(\tau) = \tilde{B}(\tau)$ for all $\tau \in \Theta$, then $A = B$ (at least when $\mathcal{H}$ consists of analytic functions, see Zhu [34]). For more details, see [6, 14, 18, 21, 26, 27]. So, the map $A \rightarrow \tilde{A}$ is injective [20]. The Berezin set (range), the Berezin number (radius), and the Berezin norm of an operator A are defined, respectively, by

$$\mathbf{Ber}(A) := \{\langle A\hat{k}_\tau, \hat{k}_\tau \rangle : \tau \in \Theta\}, \quad \mathbf{ber}(A) := \sup_{\tau \in \Theta} |\langle A\hat{k}_\tau, \hat{k}_\tau \rangle|,$$

and

$$\|A\|_{\mathbf{ber}} := \sup_{\tau, \mu \in \Theta} |\langle A\hat{k}_\tau, \hat{k}_\mu \rangle|.$$

Clearly, by the above definition, we have

$$\mathbf{ber}(A) \leq \|A\|_{\mathbf{ber}} \quad \text{for } A \in \mathcal{L}(\mathcal{H}). \quad (1.1)$$

Moreover, for $A, B \in \mathcal{L}(\mathcal{H})$, it is clear from the above definitions of the Berezin radius and the Berezin norm that the following properties hold:

- (1) $\mathbf{ber}(\alpha A) = |\alpha| \mathbf{ber}(A)$ for all $\alpha \in \mathbb{C}$;
- (2) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$;
- (3) $\|\lambda T\|_{\mathbf{ber}} = |\lambda| \|T\|_{\mathbf{ber}}$ for all $\lambda \in \mathbb{C}$;
- (4) $\|A + B\|_{\mathbf{ber}} \leq \|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}}$;
- (5) $\mathbf{ber}(A) = \mathbf{ber}(A^*)$ and $\|A\|_{\mathbf{ber}} = \|A^*\|_{\mathbf{ber}}$.

For all positive operators $A \in \mathcal{L}(\mathcal{H})$, we have $\mathbf{ber}(A) = \|A\|_{\mathbf{ber}}$, see [1].

For a bounded linear operator A on a Hilbert space $\mathcal{H}$, the numerical range $W(A)$ is defined by $W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H} \text{ \& } \|x\| = 1\}$. Moreover, the numerical radius is defined by

$$w(A) = \sup \{|\lambda| : \lambda \in W(A)\} = \sup_{\|x\|=1} |\langle Ax, x \rangle|.$$

It is clear that the Berezin transform $\tilde{A}$ is the bounded function on Θ whose values lie in the numerical range of the operator A , and hence $\mathbf{Ber}(A) \subseteq W(A)$ and $\mathbf{ber}(A) \leq w(A)$ for all $A \in \mathbb{B}(\mathcal{H})$. Karaev [28] showed that for $A = S \otimes S \in \mathbb{B}(\mathcal{H}^2)$, where S is the shift

operator defined by $Sf(z) = zf(z)$ on the Hardy-Hilbert space $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we have $\tilde{A}(\lambda) = |\lambda|^2(1 - |\lambda|^2)$, and thus $\mathbf{Ber}(A) = [0, \frac{1}{4}] \not\subseteq [0, 1] = W(A)$ and $\mathbf{ber}(A) = \frac{1}{4} < 1 = w(A)$. For more information about the Berezin number and the Berezin norm, see [3, 4, 7, 10–12, 14, 19, 23, 27, 32] and references therein.

Let us consider the finite dimensional setting. Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $\Theta = \{1, \dots, n\}$. We can consider $\mathbb{C}^n$ as the set of all functions mapping $\Theta \rightarrow \mathbb{C}$ by $z(i) = z_i$. Letting e_i be the i th standard basis vector for $\mathbb{C}^n$ under the standard inner product, we can view $\mathbb{C}^n$ as an RKHS with kernel

$$k(i, j) = \langle e_j, e_i \rangle.$$

Note that $k_i = \hat{k}_i$ for every $i = 1, \dots, n$. For any $n \times n$ complex matrix $A = (a_{ij})_{1 \leq i, j \leq n}$, we have $\langle Ae_i, e_i \rangle = a_{ii}$. Thus, the Berezin range, the Berezin radius, and the Berezin norm of A are simply by

$$\mathbf{Ber}(A) = \{a_{ii} : i = 1, \dots, n\}, \quad \mathbf{ber}(A) = \max_{1 \leq i \leq n} |a_{ii}|, \quad \text{and} \quad \|A\|_{\mathbf{ber}} = \max_{1 \leq i, j \leq n} |a_{ij}|,$$

respectively, see [15] and references therein.

A binary function σ on $[0, \infty)$ is called a mean if the following conditions are satisfied:

- (1) If $a \leq b$, then $a \leq a \sigma b \leq b$;
- (2) $a \leq c$ and $b \leq d$ imply $a \sigma b \leq c \sigma d$;
- (3) σ is continuous in both variables;
- (4) $t(a \sigma b) = (ta) \sigma (tb)$ ($t > 0$).

It is easy to see that $a \sigma a = a$, ($a \geq 0$). For instance, if $\mu \in (0, 1)$, then the weighted geometric mean is $a \sharp_\mu b = a^{1-\mu} b^\mu$, the weighted arithmetic mean is $a \nabla_\mu b = (1-\mu)a + \mu b$, and the weighted harmonic mean is $a !_\mu b = ((1-\mu)a^{-1} + \mu b^{-1})^{-1}$, with

$$a !_\mu b \leq a \sharp_\mu b \leq a \nabla_\mu b \quad \text{for all } a, b > 0.$$

A mean σ is symmetric if $a \sigma b = b \sigma a$ for all positive numbers a, b . For more information about means, see [17] and references therein.

For a symmetric mean σ , a parametrized mean σ_t , $0 \leq t \leq 1$, is called an interpolational path for σ if it satisfies

- (1) $a \sigma_0 b = a$, $a \sigma_{1/2} b = a \sigma b$, and $a \sigma_1 b = b$;
- (2) $(a \sigma_p a) \sigma (a \sigma_q a) = a \sigma_{\frac{p+q}{2}} b$ for all $p, q \in [0, 1]$;
- (3) The map $t \in [0, 1] \mapsto a \sigma_t b$ is continuous for each a and b ;
- (4) σ_t is increasing in each of its components for $t \in [0, 1]$.

It is easy to see that the set of all $r \in [0, 1]$ satisfying

$$(a \sigma_p b) \sigma_r (a \sigma_q b) = a \sigma_{rp+(1-r)q} b$$

for all p, q is a convex subset of $[0, 1]$ including 0 and 1. For instance, the power means

$$a m_r b = \left(\frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \quad (r \in [-1, 1])$$

are some typical interpolational means. Their interpolational paths are

$$a m_{r,t} b = ((1-t)a^r + tb^r)^{\frac{1}{r}} \quad (t \in [0, 1] \text{ and } r \in [-1, 1]).$$

In particular, $am_{1,t}b = a\nabla_t b$, $am_{0,t}b = \lim_{r \rightarrow 0} (am_{r,t}b) = a\sharp_t b$, and $am_{-1,t}b = a!_t b$.

Let f be a continuous real valued function defined on an interval J . It said to be operator convex on J if $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$ for all self-adjoint operators $A, B \in \mathcal{L}(\mathcal{H})$ with spectra in J and all $\lambda \in [0, 1]$

One of the most basic, yet useful inequalities, is the Cauchy–Schwarz inequality for the inner product as follows

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad \text{for all } x, y \in \mathcal{H}. \quad (1.2)$$

Buzano [13] showed an extension of the following Cauchy–Schwarz inequality

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} (|\langle x, y \rangle| + \|x\| \|y\|) \leq \|x\| \|y\| \quad \text{for all } x, y, e \in \mathcal{H}, \quad (1.3)$$

where $\|e\| = 1$. Several generalizations, refinements, and applications of the Cauchy–Schwarz inequality in various settings have been given, and some results related to integral inequalities are presented. For other results and refinements of the Cauchy–Schwarz inequality, the reader can consult [13] and references therein.

Recently, Bakherad et al. [5] presented a generalization of the refined Cauchy–Schwarz inequality as follows

$$|\langle x, y \rangle|^r \leq (|\langle x, y \rangle|^r \sigma \|x\|^r \|y\|^r) \rho (|\langle x, y \rangle|^r \tau \|x\|^r \|y\|^r) \leq \|x\| \|y\|^r \quad (1.4)$$

for all $x, y \in \mathcal{H}$, three arbitrary means σ, τ, ρ on $[0, \infty)$, and $r \geq 0$. Moreover, in the same work, they obtained a refinement of the Buzano inequality (1.3), which asserts that

$$\begin{aligned} |\langle x, e \rangle \langle e, y \rangle|^r &\leq \frac{1}{2^r} (|\langle x, y \rangle| + \|x\| \|y\|)^r \\ &\leq \frac{1}{2} \left((|\langle x, y \rangle|^r \sigma \|x\|^r \|y\|^r) \rho (|\langle x, y \rangle|^r \tau \|x\|^r \|y\|^r) + \|x\|^r \|y\|^r \right) \\ &\leq \|x\|^r \|y\|^r, \end{aligned} \quad (1.5)$$

for all $x, y, e \in \mathcal{H}$ with $\|e\| = 1$, three arbitrary means σ, τ, ρ on $[0, \infty)$, and $r \geq 1$. Furthermore, they showed

$$|\langle x, y \rangle| \leq \sqrt{(|\langle x, y \rangle|^2 \sigma \|x\| \|y\|)} \rho \sqrt{(|\langle x, y \rangle| \|x\| \|y\| \tau \|x\|^2 \|y\|^2)} \leq \|x\| \|y\|. \quad (1.6)$$

and

$$|\langle x, y \rangle| \leq \sqrt{\left[(|\langle x, y \rangle|^2 \sigma \|x\| \|y\|) \right] \rho \left[(|\langle x, y \rangle| \|x\| \|y\| \tau \|x\|^2 \|y\|^2) \right]} \leq \|x\| \|y\| \quad (1.7)$$

for all $x, y \in \mathcal{H}$, three arbitrary means σ, τ, ρ on $[0, \infty)$, and $r \geq 0$. These inequalities are also refinements of the Cauchy–Schwarz inequality.

Let us recall the arithmetic geometric mean inequality for singular values proved by Bhatia and Kittaneh [8]

$$s_i(A^*B) \leq \frac{1}{2} s_i(|A|^2 + |B|^2) \quad \text{for } i = 1, 2, \dots, n,$$

where A and B are $n \times n$ matrices and $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of A . Applying the definition of the unitarily invariant norm, we get

$$|||A^*B||| \leq \frac{1}{2} ||| |A|^2 + |B|^2 |||$$

for all $n \times n$ matrices A and B . The above inequality can be extended as follows

$$w^r(A^*B) \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \| \quad \text{for } r \geq 1.$$

for $A, B \in \mathcal{L}(\mathcal{H})$, which is associated with the numerical radius and the operator norm. There are several improvements and refinements of the above inequality, see [16] and references therein.

An analogous of the above inequality, the authors [23, Theorem 2.5] presented the following inequality

$$\mathbf{ber}^r(A^*B) \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}} \quad (1.8)$$

for all $A, B \in \mathcal{L}(\mathcal{H})$ and all $r \geq 1$, which is associated with the Berezin radius and the Berezin norm and involves the product of operators. There are several generalizations and refinements of the inequality (1.8), that have been proven recently, see [22, 25, 33]. In [22], for $r = 2$, the authors showed that

$$\begin{aligned} \mathbf{ber}^2(A^*B) &\leq \frac{t}{2t+2} \| |A|^4 + |B|^4 \|_{\mathbf{ber}} + \frac{1}{2t+2} w(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}} \\ &\leq \frac{1}{2} \| |A|^4 + |B|^4 \|_{\mathbf{ber}} \end{aligned} \quad (1.9)$$

for all $A, B \in \mathcal{L}(\mathcal{H})$ and $t \geq 0$.

In the present paper, we establish new Berezin radius upper bounds for Hilbert space operators by using a recent generalization of the refined celebrated Cauchy–Schwarz inequality. For instance, we refine the inequality (1.8) and generalize the inequality (1.9). Moreover, the obtained upper bounds have been compared with the previously known bounds to demonstrate their reliability.

2 Main results

In this section, we establish and generalize Berezin radius upper bounds for Hilbert space operators.

To prove our Berezin radius inequalities, we need several known lemmas. The first lemma is McCarthy's inequality for positive operators.

Lemma 2.1 [17] (McCarthy's inequality) *Let $A \in \mathcal{L}(\mathcal{H})$ be positive. Then for all unit vectors $x \in \mathcal{H}$, we have*

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle,$$

where $r \geq 1$. This inequality is reversed for $0 < r \leq 1$.

In the following lemma, we give the mixed Schwarz inequality, which can be found in [29].

Lemma 2.2 Let $A \in \mathcal{L}(\mathcal{H})$ and let $x, y \in \mathcal{H}$. Then

$$|\langle Ax, y \rangle|^2 \leq \langle |A|^{2s} x, x \rangle \langle |A^*|^{2(1-s)} y, y \rangle \text{ for all } 0 \leq s \leq 1.$$

The Berezin radius and the Berezin norm have several properties that are similar to the numerical radius and the operator norm, respectively. To learn about the numerical radii of operator matrices and their applications, we refer to [9, 30, 31]. The operator norm is unitarily invariant, i.e., $\|U^*AV\| = \|A\|$ for all $A \in \mathcal{L}(\mathcal{H})$ and all unitary operators $U, V \in \mathcal{L}(\mathcal{H})$. In contrast to the operator norm, the Berezin norm does not have this property. To see this, consider $A = \begin{bmatrix} 2 & 5 \\ 7 & 0 \end{bmatrix}$, $U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, and $V = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then we have

$$9.89 \approx \|U^*AV\|_{\text{ber}} \neq \|A\|_{\text{ber}} = 7.$$

In the following result, we investigate some results of the Berezin norm inequality, which are analogous of the operator norm.

Proposition 2.3 Let $A, B \in \mathcal{L}(\mathcal{H})$. Then

- (1) $\text{ber}^r(A) \leq \frac{1}{2} \| |A|^{2rs} + |A^*|^{2r(1-s)} \|_{\text{ber}}$ for $0 \leq s \leq 1$ and $r \geq 1$;
- (2) If A and B are positive and $1 \leq r \leq 2$, then $\|(A+B)^r\|_{\text{ber}} \leq 2^{r-1} \|A^r + B^r\|_{\text{ber}}$;
- (3) If A and B are positive and $0 \leq r \leq 1$, then $\|A^r + B^r\|_{\text{ber}} \leq 2^{1-r} \|(A+B)^r\|_{\text{ber}}$;
- (4) If A is positive and $r \geq 1$, then $\|A\|_{\text{ber}}^r \leq \|A^r\|_{\text{ber}}$;
- (5) If A is positive and $0 \leq r \leq 1$, then $\|A^r\|_{\text{ber}} \leq \|A\|_{\text{ber}}^r$.

Proof (1) Assume that $\tau \in \Theta$ is arbitrary. Using Lemmas 2.2 and 2.1, and the arithmetic–geometric inequality, respectively, we have

$$\begin{aligned} \left| \langle A\hat{k}_\tau, \hat{k}_\tau \rangle \right|^r &\leq \left\langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \right\rangle^{\frac{r}{2}} \left\langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \right\rangle^{\frac{r}{2}} \\ &\leq \left\langle |A|^{2rs} \hat{k}_\tau, \hat{k}_\tau \right\rangle^{\frac{1}{2}} \left\langle |B|^{2r(1-s)} \hat{k}_\tau, \hat{k}_\tau \right\rangle^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left\langle \left(|A|^{2rs} + |B|^{2r(1-s)} \right) \hat{k}_\tau, \hat{k}_\tau \right\rangle \\ &\leq \frac{1}{2} \left\| |A|^{2rs} + |A^*|^{2r(1-s)} \right\|_{\text{ber}}. \end{aligned}$$

By taking the supremum over $\tau \in \Theta$ in the above inequality, we get the desired result.

(2) If $1 \leq r \leq 2$, then the function $f(t) = t^r$ is operator convex on $[0, \infty)$, see [17, Corollary 1.16]. Then, we have

$$\left(\frac{A+B}{2} \right)^r \leq \frac{A^r + B^r}{2}$$

for positive operators A and B . Hence, for every $\tau \in \Theta$, we have

$$\left\langle \left(\frac{A+B}{2} \right)^r \hat{k}_\tau, \hat{k}_\tau \right\rangle \leq \left\langle \left(\frac{A^r + B^r}{2} \right) \hat{k}_\tau, \hat{k}_\tau \right\rangle.$$

Taking the supremum over $\tau \in \Theta$ in the above inequality, we get

$$\text{ber} \left(\left(\frac{A+B}{2} \right)^r \right) \leq \text{ber} \left(\frac{A^r + B^r}{2} \right).$$

Now, in view of the fact that for a positive operator the Berezin radius and the Berezin norm are equal, we have

$$\|(A + B)^r\|_{\text{ber}} \leq 2^{r-1} \|A^r + B^r\|_{\text{ber}}.$$

(3) It follows from the operator concavity of $f(t) = t^r$ ($0 \leq r \leq 1$) on $[0, \infty)$ [17, Corollary 1.16] and an argument similar to that used in the proof of part (2).

(4) Applying Lemma 2.1 to the positive operator A and $r \geq 1$, we have $\langle A\hat{k}_\tau, \hat{k}_\tau \rangle^r \leq \langle A^r \hat{k}_\tau, \hat{k}_\tau \rangle$ for all $\tau \in \Theta$. Hence, $\text{ber}^r(A) \leq \text{ber}(A^r)$, and so $\|A\|_{\text{ber}}^r \leq \|A^r\|_{\text{ber}}$.

(5) Applying Lemma 2.1 to the positive operator A and $0 \leq r \leq 1$, with a similar approach as in part (5), we have the desired result. $\square$

Remark 2.4 We remark that Propositions 2.3(2) is an analog of the following inequality [2]

$$|||(A + B)^r||| \leq 2^{r-1} |||A^r + B^r|||$$

for all unitarily invariant norms $||| \cdot |||$, all positive operators A and B , and $r \geq 1$. Also, it follows from the operator convexity of the function $f(t) = t^r$ for $-1 \leq r \leq 0$ on $(0, \infty)$, that the inequality in Propositions 2.3(2) holds for all positive invertible operators A and B , and $-1 \leq r \leq 0$.

Now, we obtain the next result of this section, which is a refinement of the inequality (1.8) for $1 \leq r \leq 2$.

Theorem 2.5 Let $A, B \in \mathcal{L}(\mathcal{H})$ and σ, τ, ρ be three arbitrary means on $[0, \infty)$. Then

$$\begin{aligned} \text{ber}^r(A^*B) &\leq \left(\text{ber}^r(A^*B) \sigma \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \rho \left(\text{ber}^r(A^*B) \tau \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \\ &\leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\text{ber}} \end{aligned}$$

for all $1 \leq r \leq 2$.

Proof Assume that $\hat{k}_\tau \in \mathcal{H}$ and $1 \leq r \leq 2$. Replacing x by $A\hat{k}_\tau$ and y by $B\hat{k}_\tau$ in the first inequality in (1.4), respectively, we have

$$\begin{aligned} &|\langle A^*B\hat{k}_\tau, \hat{k}_\tau \rangle|^r \\ &= |\langle B\hat{k}_\tau, A\hat{k}_\tau \rangle|^r \\ &\leq \left(|\langle B\hat{k}_\tau, A\hat{k}_\tau \rangle|^r \sigma \|A\hat{k}_\tau\|^r \|B\hat{k}_\tau\|^r \right) \rho \left(|\langle B\hat{k}_\tau, A\hat{k}_\tau \rangle|^r \tau \|A\hat{k}_\tau\|^r \|B\hat{k}_\tau\|^r \right) \\ &= \left(|\langle A^*B\hat{k}_\tau, \hat{k}_\tau \rangle|^r \sigma \|A\hat{k}_\tau\|^r \|B\hat{k}_\tau\|^r \right) \rho \left(|\langle A^*B\hat{k}_\tau, \hat{k}_\tau \rangle|^r \tau \|A\hat{k}_\tau\|^r \|B\hat{k}_\tau\|^r \right). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|A\hat{k}_\tau\| \|B\hat{k}_\tau\| &= \langle |A|^2 \hat{k}_\tau, \hat{k}_\tau \rangle^{\frac{1}{2}} \langle |B|^2 \hat{k}_\tau, \hat{k}_\tau \rangle^{\frac{1}{2}} \\ &\leq \frac{1}{2} \langle (|A|^2 + |B|^2) \hat{k}_\tau, \hat{k}_\tau \rangle \quad (\text{by the arithmetic geometric inequality}) \\ &\leq \frac{1}{2} \text{ber}(|A|^2 + |B|^2) \\ &= \frac{1}{2} \| |A|^2 + |B|^2 \|_{\text{ber}} \quad (\text{since } |A|^2 + |B|^2 \text{ is positive}). \end{aligned} \quad (2.1)$$

Hence,

$$\|A\hat{k}_\tau\|^r \|B\hat{k}_\tau\|^r \leq \frac{1}{2^r} \| |A|^2 + |B|^2 \|_{\text{ber}}^r \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\text{ber}} \quad \text{for } 1 \leq r \leq 2. \quad (2.2)$$

The above last above inequality follows from Proposition 2.3(4) and (2), respectively. Therefore, the above inequalities and the monotonicity of the means yield that

$$\begin{aligned} & |\langle A^* B \hat{k}_\tau, \hat{k}_\tau \rangle|^r \\ & \leq \left(|\langle A^* B \hat{k}_\tau, \hat{k}_\tau \rangle|^r \sigma \|A\hat{k}_\tau\|^r \|B\hat{k}_\tau\|^r \right) \rho \left(|\langle A^* B \hat{k}_\tau, \hat{k}_\tau \rangle|^r \tau \|A\hat{k}_\tau\|^r \|B\hat{k}_\tau\|^r \right) \\ & \leq \left(|\langle A^* B \hat{k}_\tau, \hat{k}_\tau \rangle|^r \sigma \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \rho \left(|\langle A^* B \hat{k}_\tau, \hat{k}_\tau \rangle|^r \tau \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \\ & \quad (\text{by the first inequality in (2.2)}) \\ & \leq \left(\text{ber}^r(A^* B) \sigma \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \rho \left(\text{ber}^r(A^* B) \tau \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \end{aligned}$$

Then, by taking the supremum over all vectors $\hat{k}_\tau \in \mathcal{H}$, we get the first inequality. For the second inequality, we have

$$\begin{aligned} & \left(\text{ber}^r(A^* B) \sigma \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \rho \left(\text{ber}^r(A^* B) \tau \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \\ & \leq \left(\text{ber}^r(A^* B) \sigma \frac{\| |A|^{2r} + |B|^{2r} \|_{\text{ber}}}{2} \right) \rho \left(\text{ber}^r(A^* B) \tau \frac{\| |A|^{2r} + |B|^{2r} \|_{\text{ber}}}{2} \right) \\ & \quad (\text{by the second inequality in (2.2)}) \\ & \leq \left(\frac{\| |A|^{2r} + |B|^{2r} \|_{\text{ber}}}{2} \sigma \frac{\| |A|^{2r} + |B|^{2r} \|_{\text{ber}}}{2} \right) \rho \left(\frac{\| |A|^{2r} + |B|^{2r} \|_{\text{ber}}}{2} \tau \frac{\| |A|^{2r} + |B|^{2r} \|_{\text{ber}}}{2} \right) \\ & \quad (\text{by the inequality 1.8}) \\ & = \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\text{ber}} \quad (\text{by the properties of the means}), \end{aligned}$$

as required. $\square$

As a special case of Theorem 2.5, we have the next result, which is a refinement of the inequality (1.8) for $1 \leq r \leq 2$.

Corollary 2.6 *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then*

$$\begin{aligned} \text{ber}^r(A^* B) & \leq \frac{1}{2^{r\mu} p} \text{ber}^{r(1-\mu)}(A^* B) \| |A|^2 + |B|^2 \|_{\text{ber}}^{r\mu} + \frac{1}{2^{r(1-\nu)} q} \text{ber}^{r\nu}(A^* B) \| |A|^2 + |B|^2 \|_{\text{ber}}^{r(1-\nu)} \\ & \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\text{ber}} \end{aligned} \quad (2.3)$$

for all real numbers $1 \leq r \leq 2$, $\mu, \nu \in [0, 1]$, and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Assume that $\mu, \nu \in [0, 1]$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. Putting $\sigma = \sharp_\mu$, $\rho = \nabla_{\frac{1}{q}}$, and $\tau = \sharp_\nu$ in Theorem 2.5, we get the desired results. $\square$

By taking $\mu = \nu = \frac{1}{2}$ in Corollary 2.6, we get the next result.

Corollary 2.7 *Let $A, B \in \mathcal{L}(\mathcal{H})$ and $1 \leq r \leq 2$. Then*

$$\mathbf{ber}^r(A^*B) \leq \frac{1}{2^{\frac{r}{2}}} \mathbf{ber}^{\frac{r}{2}}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{\frac{r}{2}} \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}}.$$

In particular, we have

$$\mathbf{ber}^r(A^2) \leq \frac{1}{2^{\frac{r}{2}}} \mathbf{ber}^{\frac{r}{2}}(A^2) \| |A|^2 + |A^*|^2 \|_{\mathbf{ber}}^{\frac{r}{2}} \leq \frac{1}{2} \| |A|^{2r} + |A^*|^{2r} \|_{\mathbf{ber}}.$$

Proof Putting $\mu = \nu = \frac{1}{2}$ in Corollary 2.6, we get the first result. Taking $B = A$ and using the fact that $\mathbf{ber}((A^*)^2) = \mathbf{ber}(A^2)$ in the first result, we get the second result. $\square$

Remark 2.8 Assume that $A, B \in \mathcal{L}(\mathcal{H})$ and $t \geq 0$. If we take $r = 2$, $\mu = 1$, $\nu = \frac{1}{2}$, and $\frac{1}{p} = \frac{t}{1+t}$ in the first inequality of (2.3), then

$$\mathbf{ber}^2(A^*B) \leq \frac{t}{2^2(1+t)} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^2 + \frac{1}{2+2t} \mathbf{ber}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}.$$

Using Proposition 2.3(4) and (3), we get

$$\begin{aligned} \mathbf{ber}^2(A^*B) &\leq \frac{t}{2^2(1+t)} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^2 + \frac{1}{2(1+t)} \mathbf{ber}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}} \\ &\leq \frac{t}{2+2t} \| |A|^4 + |B|^4 \|_{\mathbf{ber}} + \frac{1}{2+2t} \mathbf{ber}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}} \\ &\quad \text{(by the second inequality in (2.2))} \\ &\leq \frac{1}{2} \| |A|^4 + |B|^4 \|_{\mathbf{ber}} \quad \text{(by the inequality (1.9)).} \end{aligned}$$

Hence, the inequalities in (2.3) are generalizations and refinements of the inequality (1.9).

Taking the double integral over the inequalities (2.3), we get the next result, which is a refinement of the inequality (1.8).

Corollary 2.9 *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then*

$$\begin{aligned} \mathbf{ber}^r(A^*B) &\leq \int_0^1 \frac{1}{2^{r\mu}} \mathbf{ber}^{r(1-\mu)}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{r\mu} d\mu \\ &\leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}} \end{aligned}$$

for all $1 \leq r \leq 2$.

Proof Taking the double integral over the inequalities (2.3) on $[0, 1] \times [0, 1]$, we get

$$\begin{aligned} &\mathbf{ber}^r(A^*B) \\ &= \int_0^1 \int_0^1 \mathbf{ber}^r(A^*B) d\mu d\nu \\ &\leq \int_0^1 \int_0^1 \left(\frac{1}{2^{r\mu p}} \mathbf{ber}^{r(1-\mu)}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{r\mu} + \frac{1}{2^{r(1-\nu)q}} \mathbf{ber}^{r\nu}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{r(1-\nu)} \right) d\mu d\nu \\ &\leq \frac{1}{2} \int_0^1 \int_0^1 \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}} d\mu d\nu \\ &= \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}}. \end{aligned}$$

Hence,

$$\begin{aligned}
 & \mathbf{ber}^r(A^*B) \\
 & \leq \int_0^1 \frac{1}{2^{r\mu} p} \mathbf{ber}^{r(1-\mu)}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{r\mu} d\mu + \int_0^1 \frac{1}{2^{r(1-\nu)q}} \mathbf{ber}^{r\nu}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{r(1-\nu)} d\nu \\
 & = \int_0^1 \frac{1}{2^{r\mu}} \mathbf{ber}^{r(1-\mu)}(A^*B) \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{r\mu} d\mu \\
 & \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}},
 \end{aligned}$$

as required. $\square$

Remark 2.10 As a spacial case of Corollary 2.9 for $A, B \in \mathcal{L}(\mathcal{H})$ with $0 \neq \mathbf{ber}(A^*B)$ and $2\mathbf{ber}(A^*B) \neq \| |A|^2 + |B|^2 \|_{\mathbf{ber}}$, we have

$$\mathbf{ber}^r(A^*B) \leq \frac{1}{\ln\left(\frac{\| |A|^2 + |B|^2 \|_{\mathbf{ber}}^r}{2^r \mathbf{ber}^r(A^*B)}\right)} \left(\frac{\| |A|^2 + |B|^2 \|_{\mathbf{ber}}^r}{2^r} - \mathbf{ber}^r(A^*B) \right) \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}},$$

where $1 \leq r \leq 2$.

In the next theorem, we have another refinement of the inequality (1.8) as follows.

Corollary 2.11 Let $A, B \in \mathcal{L}(\mathcal{H})$ and $1 \leq r \leq 2$. Then

$$\begin{aligned}
 & \mathbf{ber}^r(A^*B) \\
 & \leq \left((1-\alpha)\mathbf{ber}^{rs}(A^*B) + \frac{\alpha}{2^{rt}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rt} \right)^{\frac{1}{t}} \\
 & \leq \left((1-\alpha)\mathbf{ber}^{rt}(A^*B) + \frac{\alpha}{2^{rt}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rt} \right)^{\frac{1}{t}} \sharp_{\nu} \left((1-\alpha)\mathbf{ber}^{rs}(A^*B) + \frac{\alpha}{2^{rs}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rs} \right)^{\frac{1}{s}} \\
 & \leq \left((1-\alpha)\mathbf{ber}^{rs}(A^*B) + \frac{\alpha}{2^{rs}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rs} \right)^{\frac{1}{s}} \\
 & \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}}
 \end{aligned}$$

for all $\alpha, \nu \in [0, 1]$ and $0 \leq t \leq s \leq 1$.

Proof First, note that if $0 \leq t \leq s \leq 1$, then $m_{\alpha,t} \leq m_{\alpha,s}$ for all $\alpha \in [0, 1]$, see [24, p. 26]. In Theorem 2.5, substituting $\sigma = m_{\alpha,t}$, $\rho = \sharp_{\nu}$, and $\tau = m_{\alpha,s}$, we get

$$\begin{aligned}
 & \mathbf{ber}^r(A^*B) \\
 & \leq \left((1-\alpha)\mathbf{ber}^r(A^*B) + \frac{\alpha}{2^{rt}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rt} \right)^{\frac{1}{t}} \quad (\text{by the inequality (1.8) for } r = 1) \\
 & = \left((1-\alpha)\mathbf{ber}^{rt}(A^*B) + \frac{\alpha}{2^{rt}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rt} \right)^{\frac{1}{t}} \sharp_{\nu} \left((1-\alpha)\mathbf{ber}^{rt}(A^*B) + \frac{\alpha}{2^{rt}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rt} \right)^{\frac{1}{t}} \\
 & \leq \left((1-\alpha)\mathbf{ber}^{rt}(A^*B) + \frac{\alpha}{2^{rt}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rt} \right)^{\frac{1}{t}} \sharp_{\nu} \left((1-\alpha)\mathbf{ber}^{rs}(A^*B) + \frac{\alpha}{2^{rs}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rs} \right)^{\frac{1}{s}} \\
 & \leq \left((1-\alpha)\mathbf{ber}^{rs}(A^*B) + \frac{\alpha}{2^{rs}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rs} \right)^{\frac{1}{s}} \sharp_{\nu} \left((1-\alpha)\mathbf{ber}^{rs}(A^*B) + \frac{\alpha}{2^{rs}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rs} \right)^{\frac{1}{s}} \\
 & = \left((1-\alpha)\mathbf{ber}^{rs}(A^*B) + \frac{\alpha}{2^{rs}} \| |A|^2 + |B|^2 \|_{\mathbf{ber}}^{rs} \right)^{\frac{1}{s}} \quad (\text{by the definition of means}) \\
 & \leq \left((1-\alpha)\mathbf{ber}^{rs}(A^*B) + \frac{\alpha}{2^s} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}}^s \right)^{\frac{1}{s}} \quad (\text{by the second inequality in (2.2)}) \\
 & \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\mathbf{ber}} \quad (\text{by the definition of means and the inequality (1.8)}),
 \end{aligned}$$

as required. $\square$

Remark 2.12 Utilizing the inequalities (1.6) and (1.7), and using the same strategy as in the proof of Theorem 2.5, we get other refinements of the inequality (1.8) as follows

$$\begin{aligned} & \text{ber}^r(A^*B) \\ & \leq \sqrt{\text{ber}^{2r}(A^*B) \sigma \left(\frac{\text{ber}^r(A^*B) \| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right)} \rho \sqrt{\left(\text{ber}^r(A^*B) \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \tau \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^{2r}}{4^r}} \\ & \leq \frac{1}{2} \| |A|^{2r} + |B|^{2r} \|_{\text{ber}} \end{aligned}$$

and

$$\begin{aligned} & \text{ber}^{2r}(A^*B) \\ & \leq \left[\text{ber}^{2r}(A^*B) \sigma \left(\frac{\text{ber}^r(A^*B) \| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \right] \rho \left[\left(\text{ber}^r(A^*B) \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^r}{2^r} \right) \tau \frac{\| |A|^2 + |B|^2 \|_{\text{ber}}^{2r}}{4^r} \right] \\ & \leq \frac{1}{4} \| |A|^{2r} + |B|^{2r} \|_{\text{ber}}^2 \end{aligned}$$

for all $A, B \in \mathcal{L}(\mathcal{H})$, three arbitrary means σ, τ, ρ on $[0, \infty)$, and $1 \leq r \leq 2$.

In the next theorem, we obtain an upper bound for the Berezin radius, which is a refinement of Proposition 2.3(1).

Theorem 2.13 Let $A \in \mathcal{L}(\mathcal{H})$ and $1 \leq r \leq 2$. Then

$$\begin{aligned} \text{ber}^{2r}(A) & \leq \left(\text{ber}^{2r}(A) \sigma \frac{\| |A|^{4s} + |A^*|^{4(1-s)} \|_{\text{ber}}^r}{2^r} \right) \rho \left(\text{ber}^{2r}(A) \tau \frac{\| |A|^{4s} + |A^*|^{4(1-s)} \|_{\text{ber}}^r}{2^r} \right) \\ & \leq \frac{1}{2} \| |A|^{4rs} + |A^*|^{4r(1-s)} \|_{\text{ber}}. \end{aligned}$$

Proof Assume that $\hat{k}_\tau \in \mathcal{H}$, $0 \leq s \leq 1$, and $1 \leq r \leq 2$. Then, utilizing Lemma 2.2, we have

$$\left| \langle A \hat{k}_\tau, \hat{k}_\tau \rangle \right|^{2r} \leq \langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \rangle^r \langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \rangle^r.$$

Now, by replacing $|\langle x, y \rangle|^r$ by $\left| \langle A \hat{k}_\tau, \hat{k}_\tau \rangle \right|^{2r}$ and $\|x\|^r \|y\|^r$ by $\langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \rangle^r \langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \rangle^r$ in the inequalities (1.4), we get

$$\begin{aligned} \left| \langle A \hat{k}_\tau, \hat{k}_\tau \rangle \right|^{2r} & \leq \left(\left| \langle A \hat{k}_\tau, \hat{k}_\tau \rangle \right|^{2r} \sigma \langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \rangle^r \langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \rangle^r \right) \\ & \quad \rho \left(\left| \langle A \hat{k}_\tau, \hat{k}_\tau \rangle \right|^{2r} \tau \langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \rangle^r \langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \rangle^r \right) \\ & \leq \langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \rangle^r \langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \rangle^r. \end{aligned} \quad (2.4)$$

Moreover, we have

$$\begin{aligned} \langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \rangle \langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \rangle & \leq \frac{1}{2} \left(\langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \rangle^2 + \langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \rangle^2 \right) \\ & \quad \text{(by the arithmetic geometric inequality)} \\ & \leq \frac{1}{2} \langle |A|^{4s} + |A^*|^{4(1-s)} \hat{k}_\tau, \hat{k}_\tau \rangle \\ & \quad \text{(by Lemma 2.1)} \end{aligned}$$

$$\leq \frac{1}{2} \left\| |A|^{4s} + |A^*|^{4(1-s)} \right\|_{\text{ber}}$$

for all $0 \leq s \leq 1$. Applying the first inequality in (2.4) and the above inequalities, we get

$$\begin{aligned} & \left| \langle A \hat{k}_\tau, \hat{k}_\tau \rangle \right|^{2r} \\ & \leq \left(\left| \langle A \hat{k}_\tau, \hat{k}_\tau \rangle \right|^{2r} \sigma \left\langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \right\rangle^r \left\langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \right\rangle^r \right) \\ & \quad \rho \left(\left| \langle A \hat{k}_\tau, \hat{k}_\tau \rangle \right|^{2r} \tau \left\langle |A|^{2s} \hat{k}_\tau, \hat{k}_\tau \right\rangle^r \left\langle |A^*|^{2(1-s)} \hat{k}_\tau, \hat{k}_\tau \right\rangle^r \right) \\ & \leq \left(\text{ber}^{2r}(A) \sigma \frac{\left\| |A|^{4s} + |A^*|^{4(1-s)} \right\|_{\text{ber}}^r}{2^r} \right) \rho \left(\text{ber}^{2r}(A) \tau \frac{\left\| |A|^{4s} + |A^*|^{4(1-s)} \right\|_{\text{ber}}^r}{2^r} \right). \end{aligned}$$

Then, by taking the supremum over all $\hat{k}_\tau \in \mathcal{H}$, we get the first inequality. Moreover,

$$\begin{aligned} & \left(\text{ber}^{2r}(A) \sigma \frac{\left\| |A|^{4s} + |A^*|^{4(1-s)} \right\|_{\text{ber}}^r}{2^r} \right) \rho \left(\text{ber}^{2r}(A) \tau \frac{\left\| |A|^{4s} + |A^*|^{4(1-s)} \right\|_{\text{ber}}^r}{2^r} \right) \\ & \leq \left(\text{ber}^{2r}(A) \sigma \frac{\left\| |A|^{4rs} + |A^*|^{4r(1-s)} \right\|_{\text{ber}}}{2} \right) \rho \left(\text{ber}^{2r}(A) \tau \frac{\left\| |A|^{4rs} + |A^*|^{4r(1-s)} \right\|_{\text{ber}}}{2} \right) \\ & \quad \text{(by the second inequality in (2.2))} \\ & \leq \left(\frac{\left\| |A|^{4rs} + |A^*|^{4r(1-s)} \right\|_{\text{ber}}}{2} \sigma \frac{\left\| |A|^{4rs} + |A^*|^{4r(1-s)} \right\|_{\text{ber}}}{2} \right) \\ & \quad \rho \left(\frac{\left\| |A|^{4rs} + |A^*|^{4r(1-s)} \right\|_{\text{ber}}}{2} \tau \frac{\left\| |A|^{4rs} + |A^*|^{4r(1-s)} \right\|_{\text{ber}}}{2} \right) \quad \text{(by Proposition (2.3)(1))} \\ & = \frac{1}{2} \left\| |A|^{4rs} + |A^*|^{4r(1-s)} \right\|_{\text{ber}}. \end{aligned}$$

This completes the proof. $\square$

In the following theorem, by using Theorem 2.13, we obtain another refinement of Proposition 2.3(1).

Corollary 2.14 *Let $A \in \mathcal{L}(\mathcal{H})$ and $1 \leq r \leq 2$. Then*

$$\begin{aligned} \text{ber}^{2r}(A) & \leq \frac{1}{2^r \mu^p} \text{ber}^{2r(1-\mu)}(A) \left\| |A|^{4s} + |A^*|^{4(1-s)} \right\|_{\text{ber}}^{r\mu} + \frac{1}{2^{rv} q} \text{ber}^{2r(1-\nu)}(A) \left\| |A|^{4s} + |A^*|^{4(1-s)} \right\|_{\text{ber}}^{rv} \\ & \leq \frac{1}{2} \left\| |A|^{4rs} + |A^*|^{4r(1-s)} \right\|_{\text{ber}} \end{aligned}$$

for all $\mu, \nu, s \in [0, 1]$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Assume that $\mu, \nu, s \in [0, 1]$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the inequalities follow from Theorem 2.13 for $\sigma = \sharp_\nu$, $\rho = \nabla_{\frac{1}{q}}$, and $\tau = \sharp_\mu$. $\square$

In the next theorem, we present an extension of Proposition 2.3(1).

Theorem 2.15 Let $A \in \mathcal{L}(\mathcal{H})$ and σ, τ, ρ be three arbitrary means on $[0, \infty)$. Then

$$\begin{aligned} \text{ber}^{2r}(A) &\leq \frac{1}{2} \left[\left(\text{ber}^r(A^2) \sigma \frac{\| |A|^2 + |A^*|^2 \|_{\text{ber}}^r}{2^r} \right) \rho \left(\text{ber}^r(A^2) \tau \frac{\| |A|^2 + |A^*|^2 \|_{\text{ber}}^r}{2^r} \right) + \frac{\| |A|^2 + |A^*|^2 \|_{\text{ber}}^r}{2^r} \right] \\ &\leq \frac{1}{2} \| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}} \end{aligned}$$

for all $1 \leq r \leq 2$.

Proof Assume that $\mu, \nu \in [0, 1]$ and $\hat{k}_\tau \in \mathcal{H}$. Applying the second inequality in (1.5) and replacing x by $A\hat{k}_\tau$, e by $\hat{k}_\tau$, and y by $A^*\hat{k}_\tau$, we get

$$\begin{aligned} &|\langle A\hat{k}_\tau, \hat{k}_\tau \rangle|^{2r} \\ &= |\langle A\hat{k}_\tau, \hat{k}_\tau \rangle \langle \hat{k}_\tau, A^*\hat{k}_\tau \rangle|^r \\ &\leq \frac{1}{2} \left[\left(|\langle A\hat{k}_\tau, A^*\hat{k}_\tau \rangle|^r \sigma \|A\hat{k}_\tau\|^r \|A^*\hat{k}_\tau\|^r \right) \rho \left(|\langle A\hat{k}_\tau, A^*\hat{k}_\tau \rangle|^r \tau \|A\hat{k}_\tau\|^r \|A^*\hat{k}_\tau\|^r \right) + \|A\hat{k}_\tau\|^r \|A^*\hat{k}_\tau\|^r \right] \\ &= \frac{1}{2} \left[\left(|\langle A^2\hat{k}_\tau, \hat{k}_\tau \rangle|^r \sigma \|A\hat{k}_\tau\|^r \|A^*\hat{k}_\tau\|^r \right) \rho \left(|\langle A^2\hat{k}_\tau, \hat{k}_\tau \rangle|^r \tau \|A\hat{k}_\tau\|^r \|A^*\hat{k}_\tau\|^r \right) + \|A\hat{k}_\tau\|^r \|A^*\hat{k}_\tau\|^r \right] \\ &\leq \frac{1}{2} \left[\left(\text{ber}^r(A^2) \sigma \|A\hat{k}_\tau\|^r \|A^*\hat{k}_\tau\|^r \right) \rho \left(\text{ber}^r(A^2) \tau \|A\hat{k}_\tau\|^r \|A^*\hat{k}_\tau\|^r \right) + \|A\hat{k}_\tau\|^r \|A^*\hat{k}_\tau\|^r \right] \\ &\leq \frac{1}{2} \left[\left(\text{ber}^r(A^2) \sigma \frac{\| |A|^2 + |A^*|^2 \|_{\text{ber}}^r}{2^r} \right) \rho \left(\text{ber}^r(A^2) \tau \frac{\| |A|^2 + |A^*|^2 \|_{\text{ber}}^r}{2^r} \right) + \frac{\| |A|^2 + |A^*|^2 \|_{\text{ber}}^r}{2^r} \right] \\ &\quad \text{(by the first inequality in (2.2))} \end{aligned}$$

Taking the supremum over all $\hat{k}_\tau \in \mathcal{H}$, we get the first inequality. For the second inequality, we have

$$\begin{aligned} &\leq \frac{1}{2} \left[\left(\text{ber}^r(A^2) \sigma \frac{\| |A|^2 + |A^*|^2 \|_{\text{ber}}^r}{2^r} \right) \rho \left(\text{ber}^r(A^2) \tau \frac{\| |A|^2 + |A^*|^2 \|_{\text{ber}}^r}{2^r} \right) + \frac{\| |A|^2 + |A^*|^2 \|_{\text{ber}}^r}{2^r} \right] \\ &\leq \frac{1}{2} \left[\left(\text{ber}^r(A^2) \sigma \frac{\| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}}}{2} \right) \rho \left(\text{ber}^r(A^2) \tau \frac{\| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}}}{2} \right) + \frac{\| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}}}{2} \right] \\ &\quad \text{(by the second inequality in (2.2))} \\ &\leq \frac{1}{2} \left[\left(\frac{\| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}}}{2} \sigma \frac{\| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}}}{2} \right) \rho \left(\frac{\| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}}}{2} \tau \frac{\| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}}}{2} \right) \right. \\ &\quad \left. + \frac{\| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}}}{2} \right] \quad \text{(by the inequality (1.8))} \\ &= \frac{1}{2} \| |A|^{2r} + |A^*|^{2r} \|_{\text{ber}} \quad \text{(by the properties of means),} \end{aligned}$$

as required. $\square$

Theorem 2.15, for $r = 1$, yields a refinement of Proposition 2.3(1) as follows.

Corollary 2.16 Let $A \in \mathcal{L}(\mathcal{H})$. Then

$$\begin{aligned} \text{ber}^2(A) &\leq \frac{1}{2\mu+1} \text{ber}^{(1-\mu)}(A^2) \| |A|^2 + |A^*|^2 \|_{\text{ber}}^\mu + \frac{1}{2^{2-\nu}} \text{ber}^\nu(A^2) \| |A|^2 + |A^*|^2 \|_{\text{ber}}^{1-\nu} + \frac{1}{4} \| |A|^2 + |A^*|^2 \|_{\text{ber}} \\ &\leq \frac{1}{2} \| |A|^2 + |A^*|^2 \|_{\text{ber}} \end{aligned}$$

for $\mu, \nu \in [0, 1]$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Assume that $\mu, \nu, s \in [0, 1]$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. The inequalities follows from Theorem 2.15 for $\sigma = \sharp_\nu$, $\rho = \nabla_{\frac{1}{q}}$, and $\tau = \sharp_\mu$. $\square$

Remark 2.17 If f is an increasing function on $[0, \infty)$, then by using the inequality (1.8) for $r = 1$, we have

$$f(\text{ber}(A^*B)) \leq f\left(\frac{\| |A|^2 + |B|^2 \|_{\text{ber}}}{2}\right) \quad \text{for all } A, B \in \mathcal{L}(\mathcal{H}). \quad (2.5)$$

Now, if we replace $|\langle x, y \rangle|^r$ by $f(\text{ber}(A^*B))$ and $\|x\|\|y\|^r$ by $f\left(\frac{\| |A|^2 + |B|^2 \|_{\text{ber}}}{2}\right)$ in the inequalities (1.4), then we get

$$\begin{aligned} & f(\text{ber}(A^*B)) \\ & \leq \left[f(\text{ber}(A^*B)) \sigma f\left(\frac{\| |A|^2 + |B|^2 \|_{\text{ber}}}{2}\right) \right] \rho \left[f(\text{ber}(A^*B)) \tau f\left(\frac{\| |A|^2 + |B|^2 \|_{\text{ber}}}{2}\right) \right] \\ & \leq f\left(\frac{\| |A|^2 + |B|^2 \|_{\text{ber}}}{2}\right) \end{aligned}$$

for all three arbitrary means σ, τ, ρ and operators $A, B \in \mathcal{L}(\mathcal{H})$. These inequalities contain other refinements and generalizations of the inequality (1.8) for $r = 1$.

Remark 2.18 Let f be a continuous real valued function defined on $[0, \infty)$. If f is increasing and convex, then Proposition 2.3(4) can be extended to

$$f(\|A\|_{\text{ber}}) \leq \|f(A)\|_{\text{ber}}$$

for all positive operators $A \in \mathcal{L}(\mathcal{H})$. Moreover if f is operator convex on $[0, \infty)$, Proposition 2.3(2) can be extended to

$$\left\| f\left(\frac{A+B}{2}\right) \right\|_{\text{ber}} \leq \left\| \frac{f(A) + f(B)}{2} \right\|_{\text{ber}}$$

for all positive operators $A, B \in \mathcal{L}(\mathcal{H})$. Now, if f is operator convex and increasing on $[0, \infty)$, then we have

$$f\left(\left\| \frac{A+B}{2} \right\|_{\text{ber}}\right) \leq \left\| \frac{f(A) + f(B)}{2} \right\|_{\text{ber}}$$

for all positive operators $A, B \in \mathcal{L}(\mathcal{H})$.

Acknowledgements The first author would like to thank the Tusi Mathematical Research Group (TMRG).

Author Contributions The authors contributed equally to the manuscript and read and approved the final manuscript.

Funding The authors declare that there is no source of funding for this research.

Data availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Aronzajn, N.: Theory of reproducing kernels. *Trans. Am. Math. Soc.* **68**, 337–404 (1950)
2. Aujla, J., Silva, F.: Weak majorization inequalities and convex functions. *Linear Algebra Appl.* **369**, 217–233 (2003)
3. Bakherad, M.: Some Berezin number inequalities for operator matrices. *Czechoslov. Math. J.* **68**(4), 997–1009 (2018)
4. Bakherad, M., Karaev, M.T.: Berezin number inequalities for Hilbert space operators. *Concr. Oper.* **6**(1), 33–43 (2019)
5. Bakherad, M., Kittaneh, F., Krnić, M.: Complementary and refined inequalities for the Cauchy–Schwarz inequality involving means, preprint
6. Bakherad, M., Lashkaripour, R., Hajmohamadi, M., Yamancı, U.: Complete refinements of the Berezin number inequalities. *J. Math. Inequal.* **13**(4), 1117–1128 (2019)
7. Bakherad, M., Yamancı, U.: New estimations for the Berezin number inequality. *J. Inequal. Appl.* **40**, 9 (2020)
8. Bhatia, R., Kittaneh, F.: On the singular values of a product of operators. *SIAM J. Matrix Anal. Appl.* **11**(2), 272–277 (1990)
9. Bhunia, P., Dragomir, S.S., Moslehian, M.S., Paul, K.: *Lectures on Numerical Radius Inequalities*. Infosys Science Foundation Series in Mathematical Sciences, Springer, Cham (2022)
10. Bhunia, P., Sen, A., Barik, S., Paul, K.: Berezin number and Berezin norm inequalities for operator matrices. *Linear Multilinear Algebra* **72**(16), 2749–2768 (2024)
11. Bhunia, P., Paul, K., Sen, A.: Inequalities involving Berezin norm and Berezin. *Complex Anal. Oper. Theory* **17**(1), 15 (2023)
12. Bhunia, P., Sen, A., Paul, K.: Development of the Berezin number inequalities. *Acta Math. Sin. (Engl. Ser.)* **39**(7), 1219–1228 (2023)
13. Buzano, M.L.: Generalizzazione della disuguaglianza di Cauchy–Schwarz. *Rend. Sem. Mat. Univ. Politech. Torino* **31**, 405–409 (1974). (Italian)
14. Chien, F., Bakherad, M., Alomari, M.W.: Refined Berezin number inequalities via superquadratic and convex functions. *Filomat* **37**(1), 265–277 (2023)
15. Cowen, C.C., Felder, Ch.: Convexity of the Berezin range. *Linear Algebra Appl.* **647**, 47–63 (2022)
16. Dragomir, S.S.: Power inequalities for the numerical radius of a product of two operators in Hilbert spaces. *Sarajevo J. Math.* **5**, 269–278 (2009)
17. Furuta, T., Mičić Hot, J., Pečarić, J., Seo, Y.: Mond Pečarić method in operator inequalities. *Zagreb* (2005)
18. Garayev, M.T., Bakherad, M., Tapdigoglu, R.: The weighted and the Davis–Wielandt Berezin number. *Oper. Matrices* **17**(2), 469–484 (2023)
19. Garayev, M.T., Gürdal, M., Saltan, S.: Hardy type inequality for reproducing kernel Hilbert space operators and related problems. *Positivity* **21**(4), 1615–1623 (2017)
20. Guillemin, V.: Toeplitz operators in n -dimensions. *Integr. Equ. Oper. Theory* **7**(2), 145–205 (1984)
21. Gürdal, V., Gürdal, M.B.: A-Davis–Wielandt–Berezin radius inequalities. *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.* **72**(1), 182–198 (2023)
22. Gürdal, M., Tapdigoglu, R.: New Berezin radius upper bounds. *Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **49**(2), 210–218 (2023)
23. Hajmohamadi, M., Lashkaripour, R., Bakherad, M.: Improvements of Berezin number inequalities. *Linear Multilinear Algebra* **68**(6), 1218–1229 (2020)
24. Hardy, G.H., Littlewood, J.E., Polya, G.: *Inequalities*, 2nd edn. Cambridge University Press, Cambridge (1988)
25. Huban, M.B., Basaran, H., Gürdal, M.: Berezin number inequalities via convex functions. *Filomat* **36**(7), 2333–2344 (2022)
26. Karaev, M.T.: On the Berezin symbol. *Zap. Nauch. Semin. POMI* **270**, 80–89 (Russian). Translated from *Zapiski Nauchnykh Seminarov POMI* 270(2003), 2135–2140 (2000)
27. Karaev, M.T.: Berezin symbol and invertibility of operators on the functional Hilbert spaces. *J. Funct. Anal.* **238**, 181–192 (2006)
28. Karaev, M.T.: Reproducing kernels and Berezin symbols techniques in various questions of operator theory. *Complex Anal. Oper. Theory* **7**(4), 983–1018 (2013)
29. Kato, T.: Notes on some inequalities for linear operators. *Math. Ann.* **125**, 208–212 (1952)
30. Kittaneh, F.: Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24**, 283–293 (1988)
31. Kittaneh, F.: A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. *Stud. Math.* **158**, 11–17 (2003)

32. Sen, A., Bhunia, P., Paul, K.: Berezin number inequalities of operators on reproducing kernel Hilbert spaces. *Rocky Mt. J. Math.* **52**(3), 1039–1046 (2022)
33. Sen, A., Bhunia, P., Paul, K.: Bounds for the Berezin number of reproducing kernel Hilbert space operators. *Filomat* **37**(6), 1741–1749 (2023)
34. Zhu, K.: *Operator Theory in Function Spaces*, 2nd edn. Marcel Dekker, New York (2007)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.