



A generalization of the Davis–Wielandt radius for operators

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Abstract

In this work, we define a generalization of the Davis–Wielandt radius of Hilbert space operators for an arbitrary norm and obtain some applications for Hilbert–Schmidt numerical radius inequalities. For an operator $T \in \mathcal{B}(\mathcal{H})$, the Davis–Wielandt radius is defined as

$$dw_N(T) = \sup_{\theta \in \mathbb{R}} \sqrt{N^2 (\Re(e^{i\theta} T)) + N^4 (\Re(e^{i\theta} |T|))}.$$

Using the generalization of the Davis–Wielandt radius, we present some properties of $dw_N(\cdot)$ and show several lower and upper bounds for $dw_N(\cdot)$. Moreover, we obtain some results for the Hilbert–Schmidt Davis–Wielandt radius inequalities.

Keywords Davis–Wielandt radius · Hilbert–Schmidt operator · Numerical radius · Weakly unitarily invariant · Schatten p -class

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. In the case when $\dim \mathcal{H} = n$, we identify $\mathcal{B}(\mathcal{H})$ with the matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field. For $T \in \mathcal{B}(\mathcal{H})$,

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let $T = \Re(T) + i\Im(T)$ be the Cartesian decomposition of T , where the Hermitian matrices $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$ are called the real and imaginary parts of T , respectively. For any compact operator $T \in \mathcal{B}(\mathcal{H})$, let $s_1(T), s_2(T), \dots$ be the singular values of T . We say that T belongs to the set of Schatten p -class $\mathcal{C}_p(\mathcal{H})$ ($p \geq 1$) if $\|T\|_p = \left(\sum_{i=1}^{\infty} s_i(T)^p\right)^{\frac{1}{p}} = (\operatorname{tr}|T|^p)^{\frac{1}{p}} < \infty$, where in $|T| = (T^*T)^{\frac{1}{2}}$. In particular, as $p = 2$, T belongs to the set of the Hilbert–Schmidt operators $\mathcal{C}_2(\mathcal{H})$, i.e., $\|T\|_2 = (\operatorname{tr} T^*T)^{1/2} < \infty$. These norms are the unitarily invariant, that is, for every $T \in \mathcal{C}_p(\mathcal{H})$ ($p \geq 1$) and unitaries $U, V \in \mathcal{B}(\mathcal{H})$, we have $\|UTV\|_p = \|T\|_p$. Also, if $T \in \mathcal{C}_p(\mathcal{H})$, then $T^* \in \mathcal{C}_p(\mathcal{H})$ and $\|T\|_p = \|T^*\|_p$.

For a bounded linear operator T on a Hilbert space $\mathcal{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \rightarrow \langle Tx, x \rangle$ associated with the operator. More precisely

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

Also, the numerical radius is defined by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

We recall that the usual operator norm of an operator T is defined by

$$\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}.$$

In [30], the author showed that

$$w(T) = \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta}T)\|,$$

which is equivalent to the above definition. Several properties and generalizations of the numerical radius have been given; see [4, 5, 9, 18, 19, 21, 31, 32].

One of the most interesting generalizations of a numerical range is the Davis–Wielandt shell, which is defined as

$$DW(S) = \{(\langle Sx, x \rangle, \langle Sx, Sx \rangle), x \in \mathcal{H}, \|x\| = 1\}$$

for any $S \in \mathcal{B}(\mathcal{H})$. Clearly, the projection of the set $DW(S)$ on the first coordinate is $W(S)$.

The Davis–Wielandt shell and its radius were introduced and described first by Davis in [14] and [15] and Wielandt [29]. In fact, the Davis–Wielandt shell $DW(S)$ gives more information about the operator S and $W(S)$. For instance, in the finite-dimensional case, Li and Poon proved [23] (see also [24]) that the geometrical shape of their Davis–Wielandt shells can completely determine the normal property of Hilbert space operators, namely, $S \in \mathfrak{M}_{n \times n}(\mathbb{C})$ is normal if and only if $DW(S)$ is a polyhedron in $\mathbb{C} \times \mathbb{R}$ identified with $\mathbb{R}^3$. Moreover, in the finite-dimensional case, the spectrum of an operator S ; $\operatorname{sp}(S)$ is finite and $DW(S)$ is always closed; cf [23, Theorem 2.3].

These conditions are no longer equivalent for an infinite-dimensional operator S ; cf [23, Example 2.5].

In [28], Lins et al. proved that if $S \in \mathfrak{M}_{n \times n}(\mathbb{C})$ is normal, then $DW(S)$ is the convex hull of the points $(\operatorname{Re}(\lambda_j), \operatorname{Im}(\lambda_j), |\lambda_j|^2)$ ($j = 1, \dots, n$), for $\lambda_j \in \operatorname{sp}(S)$. Moreover, each point $(\operatorname{Re}(\lambda_j), \operatorname{Im}(\lambda_j), |\lambda_j|^2)$ is an extreme point of $DW(S)$. In particular case, if $n = 2$ i.e., $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has eigenvalues λ_1, λ_2 , then $DW(S)$ degenerates to the line segment joining the points $(\lambda_1, |\lambda_1|^2)$ and $(\lambda_2, |\lambda_2|^2)$. So that $\dim DW(S) \leq 1$. In fact, the condition $\dim(DW(S)) \leq 1$ holds if and only if S is normal, with at most two distinct eigenvalues. Otherwise, $DW(S)$ is an ellipsoid (without its interior) centered at $(\frac{\lambda_1 + \lambda_2}{2}, \frac{1}{2} \operatorname{tr}(|S|^2))$. Also, it was proved that if $\dim(DW(S)) \geq 2$, then $DW(S)$ is always convex. A complete description of $DW(S)$ for a quadratic operator S was given in [24]. For more details, see also [25], [26] and [28].

In [29], Wielandt showed that the Davis–Wielandt shell is a useful tool for characterizing the eigenvalues of matrices in the set

$$\{U^*TU + V^*SV : U, V \in \mathfrak{M}_{n \times n} \text{ are unitary}\}$$

for given $T, S \in \mathfrak{M}_{n \times n}$.

The Davis–Wielandt radius of $S \in \mathcal{B}(\mathcal{H})$ is defined as

$$dw(S) = \sup_{\substack{x \in \mathcal{H} \\ \|x\|=1}} \left\{ \sqrt{|\langle Sx, x \rangle|^2 + \|Sx\|^4} \right\}.$$

One can easily check that $dw(S)$ is unitarily invariant, but it does not define a norm on $\mathcal{B}(\mathcal{H})$. However, it can be represented as a special case of the Euclidean operator radius as proved in [6].

As a direct consequence, one can easily observe that

$$\max \left\{ w(S), \|S\|^2 \right\} \leq dw(S) \leq \sqrt{w^2(S) + \|S\|^4} \quad (1.1)$$

for all $S \in \mathcal{B}(\mathcal{H})$. The inequalities are sharp.

In their recent elegant work [34], Zamani and Shebrawi proved several inequalities involving the Davis–Wielandt radius and the numerical radii of Hilbert space operators. Among others, they showed that

$$dw(S) \leq \sqrt[4]{w(|S|^4 + |S|^8) + 2w^2(|S|^2 S)}.$$

Other interesting results were given in the same work [34], and have been discussed and (in some cases) improved by Bhunia et al. [11], and among others, they showed

that

$$dw^2(S) \leq \| |S|^2 + |S|^4 \|,$$

and

$$dw^2(S) \leq \frac{1}{2}w(S^2 + \|S\|^2) + \|S^4\|.$$

An important property regarding the Davis–Wielandt radius of summand of two operators was also presented in [11], as follows:

$$dw(S_1 + S_2) \leq dw(S_1) + dw(S_2) + dw(S_1^*S_2 + S_2^*S_1) \quad (1.2)$$

for all $S_1, S_2 \in \mathcal{B}(\mathcal{H})$. Based on that, an upper bound for the Davis–Wielandt radius of 2×2 off-diagonal operator matrix was given as follows:

$$dw\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \|A\| \sqrt{\frac{1}{4} + \|A\|^2} + \|B\| \sqrt{\frac{1}{4} + \|B\|^2}. \quad (1.3)$$

In [6], the first author of this paper proved that $dw(S) = \sqrt{2}w(S)$ if and only if S is a self-adjoint idempotent operator. Moreover, it has been shown that

$$\frac{1}{4} \| |S|^2 + |S^*|^2 + 2|S|^4 \| \leq dw^2(S) \leq \frac{1}{2} \| |S|^2 + |S^*|^2 + 2|S|^4 \|.$$

For more details and new results concerning the Davis–Wielandt shell and the Davis–Wielandt radius of an operator, we refer the reader to [6], [12], [13], [16], [23]–[28], and [33].

2 A generalization of the Davis–Wielandt radius

A function $N : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{R}^+$ is called a norm if, for all $S, T \in \mathcal{B}(\mathcal{H})$ and all $\alpha \in \mathbb{C}$, it satisfies the following axioms:

- (1) $N(T) \geq 0$, and $N(T) = 0$ if and only if $T = 0$.
- (2) $N(\alpha T) = |\alpha|N(T)$.
- (3) $N(T + S) \leq N(T) + N(S)$.

Moreover, we say a norm $N(\cdot)$ is an algebra norm if $N(TS) \leq N(T)N(S)$. The norm $N(\cdot)$ is a self-adjoint norm if $N(T) = N(T^*)$ for all $T \in \mathcal{B}(\mathcal{H})$. A norm $N(\cdot)$ is weakly unitarily invariant, if we have $N(U^*TU) = N(T)$ for every $T \in \mathcal{B}(\mathcal{H})$ and every unitary $U \in \mathcal{B}(\mathcal{H})$.

Lemma 1 [1] *Let $N(\cdot)$ be a norm on $\mathcal{B}(\mathcal{H})$. The function $w_N : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{R}^+$ defined by*

$$w_N(T) = \sup_{\theta \in \mathbb{R}} N(\Re(e^{i\theta}T)) \quad (2.1)$$

is a norm on $\mathcal{B}(\mathcal{H})$. Moreover, the following properties hold:

- (1) The norm $w_N(\cdot)$ is self-adjoint.
- (2) If the norm $N(\cdot)$ is weakly unitarily invariant, then so is $w_N(\cdot)$.

Another definition of $w_N(\cdot)$ is $w_N(T) = \sup_{\alpha^2 + \beta^2 = 1} N(\alpha \Re T + \beta \Im T)$ which is equivalent to (2.1). Utilizing the definition of $w_N(\cdot)$ for a norm $N(\cdot)$ on $\mathcal{B}(\mathcal{H})$, we have

$$\frac{1}{2} \max \{N(T), N(T^*)\} \leq w_N(T) \leq \frac{1}{2} (N(T) + N(T^*))$$

for all $T \in \mathcal{B}(\mathcal{H})$. In particular, if $N(\cdot)$ is self-adjoint, then

$$\frac{1}{2} N(T) \leq w_N(T) \leq N(T) \quad \text{for } T \in \mathcal{B}(\mathcal{H}).$$

Moreover, if T is self-adjoint, then $w_N(T) = N(T)$. In fact

$$w_N(T) = \sup_{\theta \in \mathbb{R}} N(\Re(e^{i\theta} T)) = \sup_{\theta \in \mathbb{R}} N(T \Re(e^{i\theta})) = N(T).$$

For further information about $w_N(\cdot)$, we refer the reader to [2, 9, 31] and references therein.

For a norm on $T \in \mathcal{B}(\mathcal{H})$, the Davis–Wielandt shell is defined as

$$DW_N(T) = \left\{ \left(N(\Re(e^{i\theta} T)), N^2(\Re(e^{i\theta} |T|)) \right) : \theta \in \mathbb{R} \right\}.$$

Applying the above term, we have the following definition: a generalization of the Davis–Wielandt radius.

Definition 1 Let $N(\cdot)$ be a norm on $\mathcal{B}(\mathcal{H})$. The function $d w_N(\cdot) : \mathcal{B}(\mathcal{H}) \longrightarrow \mathbb{R}^+$ is defined as

$$d w_N(T) = \sup_{\lambda \in DW_N(T)} |\lambda| = \sup_{\theta \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\theta} T)) + N^4(\Re(e^{i\theta} |T|))}$$

for every $T \in \mathcal{B}(\mathcal{H})$.

Moreover, we define the set

$$\mathcal{AGB}(\mathcal{H}) = \left\{ N(\cdot) : N(\cdot) \text{ is an algebra norm and } N(T) = N(|T|) \text{ for all } T \in \mathcal{B}(\mathcal{H}) \right\}.$$

The operator norm and the p -Schatten norms belong to the set $\mathcal{AGB}(\mathcal{H})$. Moreover, if $N(\cdot)$ is a self-adjoint algebra norm, such that $N(U) \leq 1$ for every partial isometry $U \in \mathcal{B}(\mathcal{H})$, then for all $T \in \mathcal{B}(\mathcal{H})$, we have $N(T) = N(|T|)$. Therefore,

$N(\cdot)$ belongs to $\mathcal{AGB}(\mathcal{H})$. In fact, for a self-adjoint algebra norm $N(\cdot)$ that has the property $N(U) \leq 1$ for all partial isometry $U \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} N(T) &= N(U|T|) \leq N(U)N(|T|) \leq N(|T|) \\ &= N(TU^*) \leq N(T)N(U^*) = N(T)N(U) \leq N(T), \end{aligned}$$

where $T = U|T|$ is the polar decomposition of T . Hence, $N(T) = N(|T|)$.

Moreover, for $T \in \mathcal{M}_2(\mathbb{C})$, we define the maximum norm $\|T\|_{\max} = 2 \max_{i,j} |a_{ij}|$.

Then, for $T = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$, we have $|T| = \begin{bmatrix} \frac{3\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \end{bmatrix}$, $\|T\|_{\max} = 4$ and $\||T|\|_{\max} = 3\sqrt{2}$. Hence, the algebra norm $\|\cdot\|_{\max}$ does not belong to $\mathcal{AGM}_2(\mathbb{C})$.

In this paper, using the generalization of the Davis–Wielandt radius, we present some properties of $dw_N(\cdot)$ and show several lower and upper bounds for $dw_N(\cdot)$. Moreover, we obtain some results for the Hilbert–Schmidt Davis–Wielandt radius inequalities.

3 Some properties of a generalization of the Davis–Wielandt radius

In this section, we prove some properties of the generalization of the Davis–Wielandt radius for operators. The Davis–Wielandt radius for operators is not a norm in general, but it has the following property.

Proposition 1 *If $N(\cdot)$ is weakly unitarily invariant norm, then $dw_N(\cdot)$ is weakly unitarily invariant, that is*

$$dw_N(U^*TU) = dw_N(T)$$

for all $T, U \in \mathcal{B}(\mathcal{H})$, where U is unitary.

Proof Assume that $N(\cdot)$ is weakly unitarily invariant and $U \in \mathcal{B}(\mathcal{H})$ is unitary. Then

$$\begin{aligned} dw_N(U^*TU) &= \sup_{\theta \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\theta}U^*TU)) + N^4(\Re(e^{i\theta}U^*|T|U))} \\ &= \sup_{\theta \in \mathbb{R}} \sqrt{N^2(U^*\Re(e^{i\theta}T)U) + N^4(U^*\Re(e^{i\theta}|T|)U)} \\ &= \sup_{\theta \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\theta}T)) + N^4(\Re(e^{i\theta}|T|))} \quad (\text{by the assumption}) \\ &= dw_N(T), \end{aligned}$$

as required. □

Theorem 1 *Assume that $N(\cdot) \in \mathcal{AGB}(\mathcal{H})$. Then, $dw_N(\cdot)$ is convex, that is*

$$dw_N(\alpha S + (1 - \alpha)T) \leq \alpha dw_N(S) + (1 - \alpha)dw_N(T)$$

for all $S, T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in [0, 1]$.

Proof Let $S, T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in [0, 1]$. Then, by the properties of $N(\cdot)$ and the convexity of the function $f(x) = x^2$, we have

$$\begin{aligned} N(\Re(e^{i\theta}(\alpha S + (1 - \alpha)T))) &= N(\Re(e^{i\theta}\alpha S) + \Re(e^{i\theta}(1 - \alpha)T)) \\ &\leq \alpha N(\Re(e^{i\theta}S)) + (1 - \alpha)N(\Re(e^{i\theta}T)), \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} N^2(\Re(e^{i\theta}|\alpha S + (1 - \alpha)T|)) &= |\cos \theta|^2 N^2(|\alpha S + (1 - \alpha)T|) \\ &= |\cos \theta|^2 N^2((\alpha S + (1 - \alpha)T)) \\ &\leq |\cos \theta|^2 (\alpha N(S) + (1 - \alpha)N(T))^2 \\ &= \alpha |\cos \theta|^2 N^2(S) + (1 - \alpha) |\cos \theta|^2 N^2(T) \\ &\leq \alpha N^2(\Re(e^{i\theta}|S|)) + (1 - \alpha) N^2(\Re(e^{i\theta}|T|)). \end{aligned} \quad (3.2)$$

Moreover, the Minkowski inequality says that

$$\sqrt{(x + s)^2 + (y + z)^2} \leq \sqrt{x^2 + y^2} + \sqrt{s^2 + z^2} \quad \text{for all } x, y, s, z \in \mathbb{R}. \quad (3.3)$$

Hence, for $X = \alpha S + (1 - \alpha)T$, we have

$$\begin{aligned} dw_N(X) &= \sup_{\theta \in \mathbb{R}} \left\{ \left| \left(N(\Re(e^{i\theta}X)), N^2(\Re(e^{i\theta}|X|)) \right) \right| \right\} \\ &\leq \sup_{\theta \in \mathbb{R}} \left\{ \left| \left(\alpha N(\Re(e^{i\theta}S)) + (1 - \alpha)N(\Re(e^{i\theta}T)) \right. \right. \right. \\ &\quad \left. \left. , \alpha N^2(\Re(e^{i\theta}|S|)) + (1 - \alpha)N^2(\Re(e^{i\theta}|T|)) \right) \right| \right\} \\ &\quad \text{(by inequalities (3.1) and (3.2))} \\ &\leq \sup_{\theta \in \mathbb{R}} \left\{ \alpha \left| \left(N(\Re(e^{i\theta}S)), N^2(\Re(e^{i\theta}|S|)) \right) \right| \right\} \\ &\quad + \sup_{\theta \in \mathbb{R}} \left\{ (1 - \alpha) \left| \left(N(\Re(e^{i\theta}T)), N^2(\Re(e^{i\theta}|T|)) \right) \right| \right\} \\ &\quad \text{(by inequality (3.3))} \\ &= \alpha \sup_{\theta \in \mathbb{R}} \left(\left| \left(N(\Re(e^{i\theta}S)), N^2(\Re(e^{i\theta}|S|)) \right) \right| \right) \\ &\quad + (1 - \alpha) \sup_{\theta \in \mathbb{R}} \left(\left| \left(N(\Re(e^{i\theta}T)), N^2(\Re(e^{i\theta}|T|)) \right) \right| \right) \\ &= \alpha dw_N(S) + (1 - \alpha) dw_N(T), \end{aligned}$$

as required. $\square$

Corollary 1 Let $N(\cdot) \in \mathcal{AGB}(\mathcal{H})$. Then

$$\begin{aligned} 2\nu \left(\frac{dw_N(T) + dw_N(S)}{2} - dw_N\left(\frac{S+T}{2}\right) \right) \\ \leq (1-\lambda)dw_N(T) + \lambda dw_N(S) - dw_N(\lambda S + (1-\lambda)T) \\ \leq 2\Upsilon \left(\frac{dw_N(T) + dw_N(S)}{2} - dw_N\left(\frac{S+T}{2}\right) \right), \end{aligned}$$

for $S, T \in \mathcal{B}(\mathcal{H})$, $\lambda \in [0, 1]$, $\nu = \min\{\lambda, 1-\lambda\}$ and $\Upsilon = \max\{\lambda, 1-\lambda\}$.

Proof Using [20, Theorem 1], we have

$$\begin{aligned} 2\nu \left[\left(\frac{g(0) + g(1)}{2} \right) - g\left(\frac{1}{2}\right) \right] &\leq (1-\lambda)g(0) + \lambda g(1) - g(\lambda) \\ &\leq 2\Upsilon \left[\left(\frac{g(0) + g(1)}{2} \right) - g\left(\frac{1}{2}\right) \right], \end{aligned}$$

in which $\lambda \in [0, 1]$, $\nu = \min\{\lambda, 1-\lambda\}$ and $\Upsilon = \max\{\lambda, 1-\lambda\}$. Since the function $g(t) = dw_N(tS + (1-t)T)$ is convex on the interval $[0, 1]$, we obtain

$$\begin{aligned} 2\nu \left(\frac{dw_N(T) + dw_N(S)}{2} - dw_N\left(\frac{S+T}{2}\right) \right) \\ \leq (1-\lambda)dw_N(T) + \lambda dw_N(S) - dw_N(\lambda S + (1-\lambda)T) \\ \leq 2\Upsilon \left(\frac{dw_N(T) + dw_N(S)}{2} - dw_N\left(\frac{S+T}{2}\right) \right), \end{aligned}$$

where $\lambda \in [0, 1]$, $\nu = \min\{\lambda, 1-\lambda\}$ and $\Upsilon = \max\{\lambda, 1-\lambda\}$. □

In the following theorem, we show an estimation for $dw_N(\cdot)$.

Theorem 2 Let $N(\cdot)$ be a norm of $\mathcal{B}(\mathcal{H})$. Then

$$\max \left\{ w_N(T), w_N^2(|T|) \right\} \leq dw_N(T) \leq \sqrt{w_N^2(T) + w_N^4(|T|)}$$

for every $T \in \mathcal{B}(\mathcal{H})$.

Proof From Definition 1, we have

$$\begin{aligned} dw_N(T) &= \sup_{\theta \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\theta}T)) + N^4(\Re(e^{i\theta}|T|))} \\ &\leq \sqrt{\sup_{\theta \in \mathbb{R}} N^2(\Re(e^{i\theta}T)) + \sup_{\theta \in \mathbb{R}} N^4(\Re(e^{i\theta}|T|))} \\ &\leq \sqrt{\sup_{\theta \in \mathbb{R}} N^2(\Re(e^{i\theta}T)) + \sup_{\theta \in \mathbb{R}} |\cos \theta|^4 N^4(|T|)} \end{aligned}$$

$$\begin{aligned}
&\leq \sqrt{w_N^2(T) + N^4(|T|)} \\
&= \sqrt{w_N^2(T) + w_N^4(|T|)} \quad (\text{since } |T| \text{ is self-adjoint}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
dw_N(T) &= \sup_{\theta \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\theta}T)) + N^4(\Re(e^{i\theta}|T|))} \\
&\geq \sup_{\theta \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\theta}T))} \\
&= w_N(T), \quad (\text{By Def. of } w_N(\cdot))
\end{aligned}$$

and

$$\begin{aligned}
dw_N(T) &= \sup_{\theta \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\theta}T)) + N^4(\Re(e^{i\theta}|T|))} \\
&\geq \sup_{\theta \in \mathbb{R}} \sqrt{N^4(\Re(e^{i\theta}|T|))} \\
&= w_N^2(|T|).
\end{aligned}$$

Thus, the above two inequalities implies that $dw_N(T) \geq \max\{w_N(T), w_N^2(|T|)\}$, and this completes the proof of the theorem. $\square$

Theorem 3 Let $T \in \mathcal{B}(\mathcal{H})$ and $N(\cdot)$ be a norm. Then

$$dw_N(T) \geq \sqrt{\frac{1}{4}N^2(T) + \frac{1}{8}N^4(|T|)}. \quad (3.4)$$

Proof It is easy to see that, since

$$\begin{aligned}
dw_N(T) &= \sup_{\theta \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\theta}T)) + N^4(\Re(e^{i\theta}|T|))} \\
&\geq \sqrt{N^2(\Re(e^{i\theta}T)) + N^4(\Re(e^{i\theta}|T|))}
\end{aligned}$$

for every $\theta \in \mathbb{R}$, and by setting $\theta = 0$ and $\theta = \frac{\pi}{2}$ in the above inequality, respectively, we get

$$\begin{aligned}
dw_N(T) &\geq \sqrt{N^2(\Re(T)) + N^4(\Re(|T|))} \quad \text{and} \quad dw_N(T) \\
&\geq \sqrt{N^2(\Im(T)) + N^4(\Im(|T|))}.
\end{aligned}$$

Now, using the elementary inequalities $N^2(T) \leq 2(N^2(\Re(T)) + N^2(\Im(T)))$, for any norm $N(\cdot)$, we have

$$\begin{aligned}
2dw_N^2(T) &\geq N^2(\Re(T)) + N^4(\Re(|T|)) + N^2(\Im(T)) + N^4(\Im(|T|)) \\
&= N^2(\Re(T)) + N^2(\Im(T)) + N^4(\Re(|T|)) + N^4(\Im(|T|)) \\
&\geq \frac{1}{2}N^2(\Re(T) + i\Im(T)) + \frac{1}{4}N^4(\Re(|T|) + i\Im(|T|)) \\
&\quad - 2N^2(\Re(|T|))N^2(\Im(|T|)) \\
&= \frac{1}{2}N^2(T) + \frac{1}{4}N^4(|T|) - 2N^2(\Re(|T|))N^2(\Im(|T|)) \\
&= \frac{1}{2}N^2(T) + \frac{1}{4}N^4(|T|) \quad (\text{since } N(\Im(|T|)) = 0),
\end{aligned}$$

which implies that

$$dw_N(T) \geq \sqrt{\frac{1}{4}N^2(T) + \frac{1}{8}N^4(|T|)},$$

and this proves the desired inequality. $\square$

Utilizing Theorem 3, the next result holds true for the usual operator norm.

Corollary 2 Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$dw(T) \geq \sqrt{\frac{1}{4}\|T\|^2 + \frac{1}{8}\|T\|^4}. \quad (3.5)$$

Theorem 4 Let $T \in \mathcal{B}(\mathcal{H})$ and $N(\cdot)$ be a norm on $\mathcal{B}(\mathcal{H})$. Then

$$dw_N(T) \leq \inf_{\psi \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\psi}T)) + N^2(\Im(e^{i\psi}T)) + N^4(\Re(e^{i\psi}|T|))}.$$

Proof Note that for $\phi \in \mathbb{R}$, we have

$$\begin{aligned}
dw_N(e^{i\phi}T) &= \sup_{\theta \in \mathbb{R}} \sqrt{N^2(\Re(e^{i(\theta+\phi)}T)) + N^4(\Re(e^{i(\theta+\phi)}|T|))} \\
&= \sup_{\psi \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\psi}T)) + N^4(\Re(e^{i\psi}|T|))}, \quad (\text{since } \psi = \theta + \phi) \\
&= dw_N(T).
\end{aligned}$$

However, since

$$N(\Re(e^{i\phi}T)) = N\left(\frac{e^{i\psi}T + e^{-i\psi}T^*}{2}\right) = N(\cos \psi \Re(T) + \sin \psi \Im(T)),$$

and using the fact that $w_N(T) = \sup_{\alpha^2 + \beta^2 = 1} N(\alpha \Re T + \beta \Im T)$. Hence,

$$\begin{aligned}
dw_N^2(T) &= dw_N^2(e^{i\psi}T) \\
&= \sup_{\alpha^2+\beta^2=1} \left[N^2 \left(\alpha \Re(e^{i\psi}T) + \beta \Im(e^{i\psi}T) \right) + N^4 \left(\alpha \Re(e^{i\psi}|T|) + \beta \Im(e^{i\psi}|T|) \right) \right] \\
&= \sup_{\alpha^2+\beta^2=1} \left[\left(N(\alpha \Re(e^{i\psi}T)) + N(\beta \Im(e^{i\psi}T)) \right)^2 + N^4 \left(\alpha \Re(e^{i\psi}|T|) \right) \right] \\
&= \sup_{\alpha^2+\beta^2=1} \left[\left(|\alpha|N(\Re(e^{i\psi}T)) + |\beta|N(\Im(e^{i\psi}T)) \right)^2 + N^4(\alpha \Re(e^{i\psi}|T|)) \right] \\
&\leq \sup_{\alpha^2+\beta^2=1} \left[(\alpha^2 + \beta^2) \left(N^2(\Re(e^{i\psi}T)) + N^2(\Im(e^{i\psi}T)) \right) + N^4(\alpha \Re(e^{i\psi}|T|)) \right] \\
&\quad \text{(by the Cauchy–Schwarz inequality)} \\
&= N^2(\Re(e^{i\psi}T)) + N^2(\Im(e^{i\psi}T)) + N^4(\Re(e^{i\psi}|T|)),
\end{aligned}$$

whence

$$dw_N(T) \leq \inf_{\psi \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\psi}T)) + N^2(\Im(e^{i\psi}T)) + N^4(\Re(e^{i\psi}|T|))}.$$

□

Using the definition of $dw_N(\cdot)$, we have

$$\begin{aligned}
\max \left\{ \sqrt{w_N^2(\Re(T)) + w_N^4(|T|)}, \sqrt{w_N^2(\Im(T)) + w_N^4(|T|)} \right\} &\leq dw_N(T) \\
\text{for } T \in \mathcal{B}(\mathcal{H}).
\end{aligned}$$

An upper bound for $dw_N(\cdot)$ has been derived in the subsequent result.

Corollary 3 *Let $T \in \mathcal{B}(\mathcal{H})$ and $N(\cdot)$ be a norm on $\mathcal{B}(\mathcal{H})$. Then*

$$dw_N(T) \leq \sqrt{w_N^2(\Re(T)) + w_N^2(\Im(T)) + w_N^4(|T|)}.$$

Proof It follows from $\Re(T)$ and $\Im(T)$ are self-adjoint that:

$$\begin{aligned}
dw_N(T) &\leq \inf_{\psi \in \mathbb{R}} \sqrt{N^2(\Re(e^{i\psi}T)) + N^2(\Im(e^{i\psi}T)) + N^4(\Re(e^{i\psi}|T|))} \\
&\leq \sqrt{N^2(\Re(T)) + N^2(\Im(T)) + N^4(\Re(|T|))} \\
&= \sqrt{w_N^2(\Re(T)) + w_N^2(\Im(T)) + w_N^4(|T|)},
\end{aligned}$$

and this completes the proof. □

4 Hilbert–Schmidt Davis–Wielandt radius inequalities

In this section, we obtain some inequalities involving the Hilbert–Schmidt Davis–Wielandt radius inequalities. When $N(\cdot)$ is the Hilbert–Schmidt norm $\|\cdot\|_2$, the Hilbert–Schmidt Davis–Wielandt radius $dw_N(\cdot)$ is denoted by $dw_2(\cdot)$, which is

$$dw_2(T) = \sup_{\theta \in \mathbb{R}} \sqrt{\|\Re(e^{i\theta}T)\|_2^2 + \|\Re(e^{i\theta}|T|)\|_2^2}.$$

In the following theorem, we have an upper bound for $dw_2(\cdot)$.

Theorem 5 *Let $T \in \mathcal{C}_2(\mathcal{H})$. Then*

$$dw_2(T) \leq \sqrt{\frac{\|T\|_2^2 + |\operatorname{tr}(T^2)|}{2}} + \|T\|_2^4.$$

Proof It follows from [1, Theorem 7]:

$$w_2(T) := \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta}T)\|_2 = \frac{1}{\sqrt{2}} \sqrt{\|T\|_2^2 + |\operatorname{tr}(T^2)|},$$

and since $|T|$ is self-adjoint, it follows that $w_2(|T|) = \||T|\|_2 = \|T\|_2$. Thus

$$\begin{aligned} dw_2(T) &= \sup_{\theta \in \mathbb{R}} \sqrt{\|\Re(e^{i\theta}T)\|_2^2 + \|\Re(e^{i\theta}|T|)\|_2^4} \\ &\leq \sqrt{\sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta}T)\|_2^2 + \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta}|T|)\|_2^4} \\ &= \sqrt{w_2^2(T) + w_2^4(|T|)} \\ &= \sqrt{\frac{\|T\|_2^2 + |\operatorname{tr}(T^2)|}{2}} + \|T\|_2^4, \end{aligned}$$

and this proves the desired result. $\square$

In the next result, we obtain a lower bound for $dw_2(\cdot)$ by applying Theorem 3.

Corollary 4 *If $T \in \mathcal{C}_2(\mathcal{H})$, then*

$$\frac{1}{2} \|T\|_2 \sqrt{1 + \frac{1}{2} \|T\|_2^2} \leq dw_2(T). \quad (4.1)$$

Proof Inequality (4.1) directly follows from Theorem 3. $\square$

From this standpoint, it is of interest to formally define the upper extremity of the radius, termed as the proximal extremity, within the context of the Hilbert–Schmidt

Davis–Wielandt radius $\overline{dw}_2(\cdot)$. This definition is considered as follows:

$$\overline{dw}_2(T) = \sqrt{\frac{\|T\|_2^2 + |\operatorname{tr}(T^2)|}{2}} + \|T\|_2^4 \quad (4.2)$$

for every $T \in \mathcal{B}(\mathcal{H})$, then we have the following result:

Theorem 6 *If $T \in \mathcal{C}_2(\mathcal{H})$, then*

$$\|T\|_2 \sqrt{\frac{1}{2} + \|T\|_2^2} \leq \overline{dw}_2(T) \leq \|T\|_2 \sqrt{1 + \|T\|_2^2}. \quad (4.3)$$

Proof The first inequality is obvious, since

$$\overline{dw}_2(T) = \sqrt{\frac{\|T\|_2^2 + |\operatorname{tr}(T^2)|}{2}} + \|T\|_2^4 \geq \sqrt{\frac{\|T\|_2^2}{2}} + \|T\|_2^4.$$

The second inequality follows from (4.2), and this fact that $|\operatorname{tr}(T^2)| \leq \|T\|_2^2$. $\square$

In the following result, we state necessary and sufficient conditions for the equality cases in (4.3).

Corollary 5 *Assume $T \in \mathcal{C}_2(\mathcal{H})$ is nonzero. Then*

- (1) $\overline{dw}_2(T) = \|T\|_2 \sqrt{\frac{1}{2} + \|T\|_2^2}$ if and only if $\operatorname{tr}(T^2) = 0$.
- (2) $\overline{dw}_2(T) = \|T\|_2 \sqrt{1 + \|T\|_2^2}$ if and only if T is normal and the squares of its nonzero eigenvalues have the same argument.

Proof (1) It follows directly from (4.2).

(2) The proof is similar to the statements expressed from [1, Corollary 2]. $\square$

Remark 1 With a glance at Corollary 4 and Theorem 6, for any $T \in \mathcal{C}_2(\mathcal{H})$, the following sequence of inequalities holds:

$$\begin{aligned} \frac{1}{2} \|T\|_2 \sqrt{1 + \frac{1}{2} \|T\|_2^2} &\leq dw_2(T) \\ &\leq \overline{dw}_2(T) \\ &\leq \|T\|_2 \sqrt{1 + \|T\|_2^2}. \end{aligned}$$

In the following, we want to give inequalities for $\overline{dw}_2$ of 2×2 operators matrices defined on $\mathcal{H} \oplus \mathcal{H}$.

Theorem 7 *Let $T, S \in \mathcal{C}_2(\mathcal{H})$. Then*

$$(a) \overline{dw_2} \left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right) \leq \sqrt{\overline{dw_2^2}(T) + \overline{dw_2^2}(S) + 2 \|T\|_2^2 \|S\|_2^2}.$$

In particular, if T, S is self-adjoint, then

$$\overline{dw_2} \left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right) = \sqrt{\overline{dw_2^2}(T) + \overline{dw_2^2}(S) + 2 \|T\|_2^2 \|S\|_2^2}.$$

$$(b) \overline{dw_2} \left(\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \right) \leq \sqrt{\overline{dw_2^2}(T) + \frac{1}{2} \|S\|_2^2 + \|S\|_2^4 + 2 \|T\|_2^2 \|S\|_2^2}.$$

Proof To prove part (a), by (4.2), we have

$$\overline{dw_2} \left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right) = \sqrt{\frac{1}{2} \left\| \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right\|_2^2 + \frac{1}{2} \left| \operatorname{tr} \left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}^2 \right) \right| + \left\| \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right\|_2^4}.$$

Since

$$\left\| \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right\|_2^2 = \|T\|_2^2 + \|S\|_2^2,$$

and

$$\left| \operatorname{tr} \left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}^2 \right) \right| = |\operatorname{tr}(T^2) + \operatorname{tr}(S^2)| \leq |\operatorname{tr}(T^2)| + |\operatorname{tr}(S^2)|,$$

we have

$$\begin{aligned} & \overline{dw_2} \left(\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right) \\ & \leq \sqrt{\frac{1}{2} (\|T\|_2^2 + \|S\|_2^2) + \frac{1}{2} (|\operatorname{tr}(T^2)| + |\operatorname{tr}(S^2)|) + (\|T\|_2^2 + \|S\|_2^2)^2} \\ & = \sqrt{\frac{1}{2} (\|T\|_2^2 + \|S\|_2^2) + \frac{1}{2} (|\operatorname{tr}(T^2)| + |\operatorname{tr}(S^2)|) + \|T\|_2^4 + \|S\|_2^4 + 2 \|T\|_2^2 \|S\|_2^2} \\ & = \sqrt{\overline{dw_2^2}(T) + \overline{dw_2^2}(S) + 2 \|T\|_2^2 \|S\|_2^2}. \end{aligned}$$

Thus, (a) is established. In particular, since T, S are self-adjoint, it follows that T^2, S^2 are positive. Therefore

$$|\operatorname{tr}(T^2) + \operatorname{tr}(S^2)| = |\operatorname{tr}(T^2)| + |\operatorname{tr}(S^2)|.$$

Hence, the desired particular case is obtained. For part (b), we note that

$$\left\| \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \right\|_2^2 = \|T\|_2^2 + \|S\|_2^2 \quad \text{and} \quad \operatorname{tr} \left(\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^2 \right) = \operatorname{tr}(T^2).$$

Thus

$$\begin{aligned}\overline{dw_2} \left(\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \right) &\leq \sqrt{\frac{1}{2} (\|T\|_2^2 + \|S\|_2^2) + \frac{1}{2} |\operatorname{tr}(T^2)| + (\|T\|_2^2 + \|S\|_2^2)^2} \\ &= \sqrt{\frac{1}{2} (\|T\|_2^2 + \|S\|_2^2) + \frac{1}{2} |\operatorname{tr}(T^2)| + \|T\|_2^4 + \|S\|_2^4 + 2\|T\|_2^2 \|S\|_2^2} \\ &= \sqrt{dw_2^2(T) + \frac{1}{2} \|S\|_2^2 + \|S\|_2^4 + 2\|T\|_2^2 \|S\|_2^2},\end{aligned}$$

and the proof is completely established as required. $\square$

Theorem 8 Let $A, B, C, D \in \mathcal{C}_2(\mathcal{H})$. Then

$$\begin{aligned}\overline{dw_2^2} \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &\leq \overline{dw_2^2}(A) + \overline{dw_2^2}(D) + \|B\|_2^2 + \|C\|_2^2 + 2\|A\|_2^2 \|D\|_2^2 \quad (4.4) \\ &\quad + \left(\|B\|_2^2 + \|C\|_2^2 \right)^2 + 2 \left(\|A\|_2^2 + \|D\|_2^2 \right) \left(\|B\|_2^2 + \|C\|_2^2 \right).\end{aligned}$$

Proof Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then, by [3, Theorem 2], we have

$$\begin{aligned}\overline{dw_2^2}(T) &= w_2^2(T) + \|T\|_2^4 \\ &\leq w_2^2(A) + w_2^2(D) + \|B\|_2^2 + \|C\|_2^2 + \left(\|A\|_2^2 + \|B\|_2^2 + \|C\|_2^2 + \|D\|_2^2 \right)^2 \\ &= w_2^2(A) + \|A\|_2^4 + w_2^2(D) + \|D\|_2^4 + \|B\|_2^2 + \|C\|_2^2 + 2\|A\|_2^2 \|D\|_2^2 \\ &\quad + \left(\|B\|_2^2 + \|C\|_2^2 \right)^2 + 2 \left(\|A\|_2^2 + \|D\|_2^2 \right) \left(\|B\|_2^2 + \|C\|_2^2 \right) \\ &= \overline{dw_2^2}(A) + \overline{dw_2^2}(D) + \|B\|_2^2 + \|C\|_2^2 + 2\|A\|_2^2 \|D\|_2^2 \\ &\quad + \left(\|B\|_2^2 + \|C\|_2^2 \right)^2 + 2 \left(\|A\|_2^2 + \|D\|_2^2 \right) \left(\|B\|_2^2 + \|C\|_2^2 \right),\end{aligned}$$

which completes the proof of the theorem 8. $\square$

Remark 2 By putting $B = C = 0$ in Theorem 8, we get

$$\overline{dw_2^2} \left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right) \leq \sqrt{dw_2^2(A) + dw_2^2(D) + 2\|A\|_2^2 \|D\|_2^2}.$$

In particular

$$\overline{dw_2} \left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \right) \leq \sqrt{2} \cdot \sqrt{dw_2^2(A) + \|A\|_2^4}.$$

Similarly, by putting $A = D = 0$ in Theorem 8, we get

$$\overline{dw_2} \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \sqrt{\|B\|_2^2 + \|C\|_2^2 + (\|B\|_2^2 + \|C\|_2^2)^2}.$$

Also, for $C = B$ and $D = A$ in Theorem 8, we have

$$\begin{aligned} & \overline{dw_2} \left(\begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) \\ & \leq \sqrt{dw_2^2(A) + \overline{dw_2^2}(A) + 2\|B\|_2^2(1 + 2\|B\|_2^4) + 2\|A\|_2^2(\|A\|_2^2 + 4\|B\|_2^2)}. \end{aligned}$$

In particular

$$\overline{dw_2} \left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right) \leq \sqrt{2}\|B\|_2 \sqrt{1 + 2\|B\|_2^2}.$$

A particular and interesting case of dw_N is when $N(\cdot) = \|\cdot\|_p$ ($1 \leq p < \infty$). In this case for $T \in \mathcal{C}_p(\mathcal{H})$, we have

$$dw_p(T) = \sup_{\theta \in \mathbb{R}} \sqrt{\|\Re(e^{i\theta}T)\|_p^2 + \|\Re(e^{i\theta}|T|)\|_p^4}. \quad (4.5)$$

Using Theorem 2, we have the following inequality:

$$\max \{w_p(T), \|T\|_p^2\} \leq dw_p(T) \leq \sqrt{w_p^2(T) + \|T\|_p^4}. \quad (4.6)$$

It is worth mentioning that the Davis–Wielandt radius dw_p was extensively studied and defined in [6]. However, it is important to recognize that this was done using a different context and approach. Other closely related results concerning $w_p(\cdot)$ have been considered in [10] and [17].

Remark 3 Note that using the classical McCarthy–Clarkson inequality for $\mathcal{C}_p(\mathcal{H})$, for any $\theta \in \mathbb{R}$ and any $1 \leq p \leq 2$, we have [22]

$$2^p \|T\|_p^p \leq 2^p (\|\Re(e^{i\theta}T)\|_p^p + \|\Im(e^{i\theta}T)\|_p^p) \leq 2^{p+1} w_p^p(T).$$

Therefore, for $p = 2$, we have an upper bound for $\overline{dw_p}(T) = \sqrt{w_p^2(T) + \|T\|_p^4}$ as follows:

$$\overline{dw_2}(T) \leq w_2(T) \sqrt{1 + 4w_2^2(T)}.$$

For more recent inequalities involving Clarkson's inequality, see [7] and [8].

Remark 4 Finally, it is imperative to draw the reader's focus toward the potential expansion of the ideas presented in this work within a broader framework of complex linear spaces, characterized by the presence of a Hermitian sesquilinear form denoted as $[\cdot, \cdot]$. By exploring these concepts in such a setting, we can delve deeper into their applicability and uncover new insights that transcend the confines of our current discussion.

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