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SOME BEREZIN NUMBER INEQUALITIES FOR OPERATOR MATRICES

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Abstract. The Berezin symbol $\tilde{A}$ of an operator A acting on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ over some (nonempty) set is defined by $\tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle$, $\lambda \in \Omega$, where $\hat{k}_\lambda = k_\lambda/\|k_\lambda\|$ is the normalized reproducing kernel of $\mathcal{H}$. The Berezin number of the operator A is defined by $\mathbf{ber}(A) = \sup\limits_{\lambda\in\Omega} |\tilde{A}(\lambda)| = \sup\limits_{\lambda\in\Omega} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|$. Moreover, $\mathbf{ber}(A) \leqslant w(A)$ (numerical radius). We present some Berezin number inequalities. Among other inequalities, it is shown that if $\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$, then

$$\mathbf{ber}(\mathbf{T}) \leqslant \frac{1}{2}(\mathbf{ber}(A) + \mathbf{ber}(D)) + \frac{1}{2}\sqrt{(\mathbf{ber}(A) - \mathbf{ber}(D))^2 + (\|B\| + \|C\|)^2}.$$

Keywords: reproducing kernel; Berezin number; numerical radius; operator matrix

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ with the identity I. In the case when $\dim\mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices having entries in the complex field. An operator $A \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathcal{H}$, and then we write $A \geqslant 0$. Let $r(\cdot)$ denote the spectral radius. The numerical range and numerical radius of $A \in \mathbb{B}(\mathcal{H})$ are defined by

$$W(A) := \{\langle Ax, x\rangle \colon\ x \in \mathcal{H}, \|x\| = 1\} \quad \text{and} \quad w(A) := \sup\{|\lambda| \colon\ \lambda \in W(A)\},$$

respectively. It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathbb{B}(\mathcal{H})$, we have $\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|$

(see [4], page 9). For further information about numerical radius we refer the reader to [1], [5], [15].

A functional Hilbert space is a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on a (nonempty) set Ω, which has the property that point evaluations are continuous, i.e., for each $\lambda \in \Omega$ the map $f \to f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. Then the Riesz representation theorem ensures that for each $\lambda \in \Omega$ there is a unique element k_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The collection $\{k_\lambda \colon \lambda \in \Omega\}$ is called the reproducing kernel of $\mathcal{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by [6], Problem 37:

$$k_\lambda(z) = \sum_n \overline{e_n(\lambda)}\, e_n(z).$$

For $\lambda \in \Omega$, let $\hat{k}_\lambda = k_\lambda / \|k_\lambda\|$ be the normalized reproducing kernel of $\mathcal{H}$. For a bounded linear operator A on $\mathcal{H}$, the function $\tilde{A}$ defined on Ω by $\tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$ is the Berezin symbol of A, which was first introduced by Berezin [2], [3]. Berezin set and Berezin number of the operator A are defined by (see [11])

$$\mathbf{Ber}(A) := \{\tilde{A}(\lambda) \colon \lambda \in \Omega\} \quad \text{and} \quad \mathbf{ber}(A) := \sup\{|\tilde{A}(\lambda)| \colon \lambda \in \Omega\},$$

respectively. It is clear that the Berezin symbol $\tilde{A}$ is a bounded function on Ω whose values lie in the numerical range of the operator A and hence

$$\mathbf{Ber}(A) \subseteq W(A) \quad \text{and} \quad \mathbf{ber}(A) \leqslant w(A)$$

for all $A \in \mathbb{B}(\mathcal{H})$. Karaev in [12] showed that if $\mathcal{H}^2$ is the Hardy space, then we take $A = \langle \cdot\,, z\rangle z$ in $\mathcal{H}^2$; an elementary calculation shows that $\tilde{A}(\lambda) = |\lambda|^2(1 - |\lambda|^2)$, and thus

$$\mathbf{Ber}(A) = \left[0, \frac{1}{4}\right] \subsetneq [0,1] = W(A) \quad \text{and} \quad \mathbf{ber}(A) = \frac{1}{4} \lneq 1 = w(A).$$

Moreover, Berezin number of an operator A satisfies the following properties:

(a) $\mathbf{ber}(\alpha A) = |\alpha|\,\mathbf{ber}(A)$ for all $\alpha \in \mathcal{C}$.

(b) $\mathbf{ber}(A + B) \leqslant \mathbf{ber}(A) + \mathbf{ber}(B)$.

Let $T_i \in \mathbb{B}(\mathcal{H}(\Omega))$, $1 \leqslant i \leqslant n$. Then we define the generalized Euclidean Berezin number of $T_1, \ldots, T_n$ as

$$\mathbf{ber}_p(T_1, \ldots, T_n) := \sup_{\lambda \in \Omega} \left(\sum_{i=1}^n |\langle T_i \hat{k}_\lambda, \hat{k}_\lambda \rangle|^p \right)^{1/p}.$$

The generalized Euclidean Berezin number $\mathbf{ber}_p$, $p \geqslant 1$, has the following properties:

(a) $\mathbf{ber}_p(\alpha T_1, \ldots, \alpha T_n) = |\alpha|\ \mathbf{ber}_p(T_1, \ldots, T_n)$ for all $\alpha \in \mathcal{C}$;

(b) $\mathbf{ber}_p(T_1 + S_1, \ldots, T_n + S_n) \leqslant \mathbf{ber}_p(T_1, \ldots, T_n) + \mathbf{ber}_p(S_1, \ldots, S_n)$,

where $T_i, S_i \in \mathbb{B}(\mathcal{H}(\Omega))$, $1 \leqslant i \leqslant n$.

The Berezin symbol has been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it has wide applications in various questions of analysis and uniquely determines the operator (i.e., $\tilde{A}(\lambda) = \widetilde{B}(\lambda)$ for all $\lambda \in \Omega$ implies $A = B$). For further information about Berezin symbol we refer the reader to [9], [10], [12], [14], [16] and references therein.

Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ be Hilbert spaces, and consider the direct sum $\mathcal{H} = \bigoplus\limits_{j=1}^{n} \mathcal{H}_j$. With respect to this decomposition, every operator $T \in \mathbb{B}(\mathcal{H})$ has an $n \times n$ operator matrix representation $T = [T_{ij}]$ with entries $T_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$, the space of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$. Let $A \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_1)$, $B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$, $C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_2)$. The operator matrix $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ is called the diagonal part of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ is the off-diagonal part. Operator matrices provide a usual tool for studying Hilbert space operators, which have been extensively studied in the literature. Hou et al. in [8] and Omer et al. in [1] established useful estimates for the spectral radius, the numerical radius, and the operator norm of an $n \times n$ operator matrix $\mathbf{T} = [T_{ij}]$. In particular, they proved that

$$r(\mathbf{T}) \leqslant r([\|T_{ij}\|]), \quad w(\mathbf{T}) \leqslant w([\|T_{ij}\|]), \quad \|\mathbf{T}\| \leqslant \|[\|T_{ij}\|]\|$$

and $w(\mathbf{T}) \leqslant w\left([t_{ij}]\right)$, where

$$t_{ij} = \begin{cases} w(T_{ij}) & \text{if } i = j, \\ \|T_{ij}\| & \text{if } i \neq j. \end{cases}$$

The Berezin number is named in honor of Felix Berezin, who introduced this concept in [2]. In this paper, we establish some inequalities involving the Berezin number of operators. By using the ideas of [1], [15] we give several upper bounds for the Berezin number and the generalized Euclidean Berezin number of Hilbert space operators.

2. Main results

Now we are in a position to present our first result.

Theorem 2.1. *Let* $\mathbf{T} = [T_{ij}]$ *be an* $n \times n$ *operator matrix with* $T_{ij} \in \mathbb{B}(\mathcal{H}(\Omega_j), \mathcal{H}(\Omega_i))$, $1 \leqslant i, j \leqslant n$. *Then*

$$\mathbf{ber}(\mathbf{T}) \leqslant w\left([t_{ij}]\right),$$

where $t_{ij} = \begin{cases} \mathbf{ber}(T_{ij}) & \text{if } i = j, \\ \|T_{ij}\| & \text{if } i \neq j. \end{cases}$

Proof. Let $\mathcal{H} = \bigoplus_{i=1}^{n} \mathcal{H}(\Omega_i)$. For every $(\lambda_1, \ldots, \lambda_n) \in \Omega_1 \times \ldots \times \Omega_n$, let $\hat{\mathbf{k}}_{(\lambda_1,\ldots,\lambda_n)} = \begin{bmatrix} k_{\lambda_1} \\ \vdots \\ k_{\lambda_n} \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}$. Then

$$\begin{aligned}
|\widetilde{\mathbf{T}}(\lambda_1, \ldots, \lambda_n)| &= |\langle \mathbf{T}\hat{\mathbf{k}}_{(\lambda_1,\ldots,\lambda_n)}, \hat{\mathbf{k}}_{(\lambda_1,\ldots,\lambda_n)} \rangle| = \left| \sum_{i,j=1}^{n} \langle T_{ij} k_{\lambda_j}, k_{\lambda_i} \rangle \right| \\
&\leqslant \sum_{i,j=1}^{n} |\langle T_{ij} k_{\lambda_j}, k_{\lambda_i} \rangle| = \sum_{i=1}^{n} |\langle T_{ii} k_{\lambda_i}, k_{\lambda_i} \rangle| + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} |\langle T_{ij} k_{\lambda_j}, k_{\lambda_i} \rangle| \\
&\leqslant \sum_{i=1}^{n} \mathbf{ber}(T_{ii}) \|k_{\lambda_i}\|^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \|T_{ij}\| \|k_{\lambda_j}\| \|k_{\lambda_i}\| \\
&= \sum_{i,j=1}^{n} t_{ij} \|k_{\lambda_j}\| \|k_{\lambda_i}\| = \langle [t_{ij}] y, y \rangle,
\end{aligned}$$

where $y = \begin{bmatrix} \|k_{\lambda_1}\| \\ \vdots \\ \|k_{\lambda_n}\| \end{bmatrix}$. It follows from $\|y\| = 1$ that $|\widetilde{\mathbf{T}}(\lambda_1, \ldots, \lambda_n)| \leqslant w([t_{ij}])$. Hence

$$\mathbf{ber}(\mathbf{T}) = \sup_{(\lambda_1,\ldots,\lambda_n) \in \Omega_1 \times \ldots \times \Omega_n} |\widetilde{\mathbf{T}}(\lambda_1, \ldots, \lambda_n)| \leqslant w([t_{ij}])$$

as required. □

Corollary 2.2. *If* $\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$, *then*

$$\mathbf{ber}(\mathbf{T}) \leqslant \frac{1}{2}(\mathbf{ber}(A) + \mathbf{ber}(D)) + \frac{1}{2}\sqrt{(\mathbf{ber}(A) - \mathbf{ber}(D))^2 + (\|B\| + \|C\|)^2}.$$

In particular, for $\mathbf{T}=\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1)\oplus\mathcal{H}(\Omega_2))$ *we have*

$$\mathbf{ber}(\mathbf{T}) \leqslant \max\{\mathbf{ber}(A), \mathbf{ber}(D)\}\,. \tag{2.1}$$

P r o o f. Using Theorem 2.1 we get the inequality

$$\begin{aligned}
\mathbf{ber}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &\leqslant w\left(\begin{bmatrix} \mathbf{ber}\,A & \|B\| \\ \|C\| & \mathbf{ber}\,D \end{bmatrix}\right) \\
&\leqslant r\left(\begin{bmatrix} \mathbf{ber}\,A & \frac{1}{2}(\|B\|+\|C\|) \\ \frac{1}{2}(\|B\|+\|C\|) & \mathbf{ber}\,D \end{bmatrix}\right) \qquad \text{(by [7], page 44)} \\
&= \frac{1}{2}(\mathbf{ber}(A)+\mathbf{ber}(D)) + \frac{1}{2}\sqrt{(\mathbf{ber}(A)-\mathbf{ber}(D))^2+(\|B\|+\|C\|)^2}.
\end{aligned}$$

In particular, if $\mathbf{T}=\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1)\oplus\mathcal{H}(\Omega_2))$, then

$$\begin{aligned}
\mathbf{ber}(\mathbf{T}) &\leqslant \frac{1}{2}(\mathbf{ber}(A)+\mathbf{ber}(D)) + \frac{1}{2}\sqrt{(\mathbf{ber}(A)-\mathbf{ber}(D))^2} \\
&= \frac{1}{2}(\mathbf{ber}(A)+\mathbf{ber}(D)) + \frac{1}{2}|\,\mathbf{ber}(A)-\mathbf{ber}(D)| \\
&= \max\{\mathbf{ber}(A), \mathbf{ber}(D)\}.
\end{aligned}$$

□

We need the following lemmas for our results. The next lemma follows from the spectral theorem for positive operators and the Jensen inequality; see [13].

Lemma 2.3 (The McCarty inequality). *Let* $T\in\mathbb{B}(\mathcal{H})$, $T\geqslant 0$ *and* $x\in\mathcal{H}$ *such that* $\|x\|\leqslant 1$. *Then*

$$\langle Tx, x\rangle^r \leqslant \langle T^r x, x\rangle$$

for $r\geqslant 1$.

P r o o f. Let $r\geqslant 1$ and $x\in\mathcal{H}$ such that $\|x\|\leqslant 1$. Fix $u=x/\|x\|$. Using the McCarty inequality we have $\langle Tu,u\rangle^r\leqslant\langle T^r u,u\rangle$, whence

$$\begin{aligned}
\langle Tx, x\rangle^r &\leqslant \|x\|^{2r-2}\langle T^r x, x\rangle \\
&\leqslant \langle T^r x, x\rangle \qquad \text{since } \|x\|\leqslant 1 \text{ and } 2r-2\geqslant 0.
\end{aligned}$$

Hence, we get the desired result. □

Lemma 2.4 ([13], Theorem 1). *Let $T \in \mathbb{B}(\mathcal{H})$ and let $x, y \in \mathcal{H}$ be any vectors. If f, g are nonnegative continuous functions on $[0, \infty)$ which satisfy the relation $f(t)g(t) = t$, $t \in [0, \infty)$, then*

$$|\langle Tx, y\rangle|^2 \leqslant \langle f^2(|T|)x, x\rangle \, \langle g^2(|T^*|)y, y\rangle,$$

in which $|T| = (T^*T)^{1/2}$.

Theorem 2.5. *Let $\mathbf{T} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$, $r \geqslant 1$, and let f, g be nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$, $t \in [0, \infty)$. Then*

$$\mathbf{ber}^r(\mathbf{T}) \leqslant 2^{r-2}\, \mathbf{ber}^{1/2}(f^{2r}(|X|) + g^{2r}(|Y^*|))\, \mathbf{ber}^{1/2}(f^{2r}(|Y|) + g^{2r}(|X^*|)).$$

Proof. For every $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2)$ (i.e., $\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2 = 1$). Then

$$\begin{aligned}
|\widetilde{\mathbf{T}}(\lambda_1, \lambda_2)|^r &= |\langle \mathbf{T}\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}\rangle|^r \\
&= |\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle + \langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r \\
&\leqslant (|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|)^r \qquad \text{(by the triangular inequality)} \\
&\leqslant \frac{2^r}{2}(|\langle Xk_{\lambda_2}, k_{\lambda_1}\rangle|^r + |\langle Yk_{\lambda_1}, k_{\lambda_2}\rangle|^r) \qquad \text{(by the convexity } f(t) = t^r\text{)} \\
&\leqslant \frac{2^r}{2}((\langle f^2(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle^{1/2}\langle g^2(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle^{1/2})^r \\
&\quad + (\langle f^2(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle^{1/2}\langle g^2(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle^{1/2})^r) \qquad \text{(by Lemma 2.4)} \\
&\leqslant \frac{2^r}{2}(\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle^{1/2}\langle g^{2r}|X^*|k_{\lambda_1}, k_{\lambda_1}\rangle^{1/2} \\
&\quad + \langle f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle^{1/2}\langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle^{1/2}) \qquad \text{(by Lemma 2.3)} \\
&\leqslant \frac{2^r}{2}(\langle f^{2r}(|X|)k_{\lambda_2}, k_{\lambda_2}\rangle + \langle g^{2r}(|Y^*|)k_{\lambda_2}, k_{\lambda_2}\rangle)^{1/2}(\langle f^{2r}(|Y|)k_{\lambda_1}, k_{\lambda_1}\rangle \\
&\quad + \langle g^{2r}(|X^*|)k_{\lambda_1}, k_{\lambda_1}\rangle)^{1/2} \qquad \text{(by the Cauchy-Schwarz inequality)} \\
&= \frac{2^r}{2}\langle (f^{2r}(|X|) + g^{2r}(|Y^*|))k_{\lambda_2}, k_{\lambda_2}\rangle^{1/2}\langle (f^{2r}(|Y|) + g^{2r}(|X^*|))k_{\lambda_1}, k_{\lambda_1}\rangle^{1/2} \\
&\leqslant \frac{2^r}{2}\, \mathbf{ber}^{1/2}(f^{2r}(|X|) + g^{2r}(|Y^*|))\, \mathbf{ber}^{1/2}(f^{2r}(|Y|) + g^{2r}(|X^*|))\|k_{\lambda_1}\|\|k_{\lambda_2}\| \\
&\leqslant \frac{2^r}{2}\, \mathbf{ber}^{1/2}(f^{2r}(|X|) + g^{2r}(|Y^*|))\, \mathbf{ber}^{1/2}(f^{2r}(|Y|) + g^{2r}(|X^*|)) \\
&\quad \times \frac{\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2}{2} \qquad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{2^r}{4}\, \mathbf{ber}^{1/2}(f^{2r}(|X|) + g^{2r}(|Y^*|))\, \mathbf{ber}^{1/2}(f^{2r}(|Y|) + g^{2r}(|X^*|)).
\end{aligned}$$

Now, taking the supremum over all $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$ we get the desired result. $\square$

Theorem 2.5 includes a special case as follows.

Corollary 2.6. *Let* $\mathbf{T} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, $0 \leqslant p \leqslant 1$ *and* $r \geqslant 1$. *Then*

$$\mathbf{ber}^r(\mathbf{T}) \leqslant 2^{r-2}\,\mathbf{ber}^{1/2}(|X|^{2rp} + |Y^*|^{2r(1-p)})\,\mathbf{ber}^{1/2}(|Y|^{2rp} + |X^*|^{2r(1-p)}).$$

Proof. The result follows immediately from Theorem 2.5 for $f(t) = t^p$ and $g(t) = t^{1-p}$, $0 \leqslant p \leqslant 1$. $\square$

Theorem 2.7. *Let* $A, B, X \in \mathbb{B}(\mathcal{H}(\Omega))$, $r \geqslant 1$ *and let* f, g *be nonnegative continuous functions on* $[0,\infty)$ *satisfying the relation* $f(t)g(t) = t$, $t \in [0,\infty)$. *Then*

$$\mathbf{ber}^r(A^*XB) \leqslant \mathbf{ber}\Big(\frac{1}{p}[B^*f^2(|X|)B]^{rp/2} + \frac{1}{q}[A^*g^2(|X^*|)A]^{rq/2}\Big),$$

where $1/p + 1/q = 1$ *and* $pr \geqslant qr \geqslant 2$.

Proof. For every $\lambda \in \Omega$, let $\hat{\mathbf{k}}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then

$$\begin{aligned}
|\langle A^*XB\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle\|^r &= |\langle XB\hat{\mathbf{k}}_\lambda, A\hat{\mathbf{k}}_\lambda\rangle|^r \\
&\leqslant (\langle f^2(|X|)B\hat{\mathbf{k}}_\lambda, B\hat{\mathbf{k}}_\lambda\rangle\langle g^2(|X^*|)A\hat{\mathbf{k}}_\lambda, A\hat{\mathbf{k}}_\lambda\rangle)^{r/2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.4)} \\
&\leqslant \frac{1}{p}\langle f^2(|X|)B\hat{\mathbf{k}}_\lambda, B\hat{\mathbf{k}}_\lambda\rangle^{rp/2} + \frac{1}{q}\langle g^2(|X^*|)A\hat{\mathbf{k}}_\lambda, A\hat{\mathbf{k}}_\lambda\rangle^{rq/2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Young's inequaliy)} \\
&= \frac{1}{p}\langle B^*f^2(|X|)B\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle^{rp/2} + \frac{1}{q}\langle A^*g^2(|X^*|)A\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle^{rq/2} \\
&\leqslant \frac{1}{p}\langle [B^*f^2(|X|)B]^{rp/2}\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle + \frac{1}{q}\langle [A^*g^2(|X^*|)A]^{rq/2}\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.3)} \\
&= \Big\langle\Big(\frac{1}{p}[B^*f^2(|X|)B]^{rp/2} + \frac{1}{q}[A^*g^2(|X^*|)A]^{rq/2}\Big)\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\Big\rangle \\
&\leqslant \mathbf{ber}\Big(\frac{1}{p}[B^*f^2(|X|)B]^{rp/2} + \frac{1}{q}[A^*g^2(|X^*|)A]^{rq/2}\Big),
\end{aligned}$$

whence

$$\begin{aligned}
\mathbf{ber}^r(A^*XB) &= \sup_{\lambda\in\Omega} |\langle A^*XB\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle|^r \\
&\leqslant \mathbf{ber}\Big(\frac{1}{p}[B^*f^2(|X|)B]^{rp/2} + \frac{1}{q}[A^*g^2(|X^*|)A]^{rq/2}\Big).
\end{aligned}$$

$\square$

Remark 2.8. In Theorem 2.7, if we add the hypothesis of contraction for operators A and B (i.e., $A^*A \leqslant I$ and $B^*B \leqslant I$) then by using Lemma 2.3 and in a fashion similar to the proof of Theorem 2.7 we get the inequality

$$\mathbf{ber}^r(A^*XB) \leqslant \mathbf{ber}\Big(\frac{1}{p}B^*f^{rp}(|X|)B + \frac{1}{q}A^*g^{rq}(|X^*|)A\Big), \tag{2.2}$$

where $r \geqslant 1$, $1/p + 1/q = 1$ are such that $pr \geqslant qr \geqslant 2$ and f, g are nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$, $t \in [0,\infty)$.

The next result follows from Theorem 2.7 and inequality (2.2) for $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $0 \leqslant \alpha \leqslant 1$.

Corollary 2.9. *Let $A, B, X \in \mathbb{B}(\mathcal{H}(\Omega))$, let $r \geqslant 1$, $1/p + 1/q = 1$ be such that $pr \geqslant qr \geqslant 2$ and $0 \leqslant \alpha \leqslant 1$. Then*

$$\mathbf{ber}^r(A^*XB) \leqslant \mathbf{ber}\Big(\frac{1}{p}[B^*|X|^{2\alpha}B]^{rp/2} + \frac{1}{q}[A^*|X^*|^{2(1-\alpha)}A]^{rq/2}\Big).$$

In particular, if A and B are contractions, then

$$\mathbf{ber}^r(A^*XB) \leqslant \mathbf{ber}\Big(\frac{1}{p}B^*|X|^{rp\alpha}B + \frac{1}{q}A^*|X^*|^{rp(1-\alpha)}A\Big).$$

Now, we need the following lemma for the next result.

Lemma 2.10. *Let $X, Y \in \mathbb{B}(\mathcal{H}(\Omega))$. If $\mathbf{ber}\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) \leqslant \mathbf{ber}\left(\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}\right)$, then $\mathbf{ber}(X) \leqslant \mathbf{ber}(Y)$.*

Proof. For every $\lambda \in \Omega$, let $\hat{\mathbf{k}}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then

$$\begin{aligned} |\langle X\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle| &= \left|\left\langle \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\mathbf{k}}_\lambda \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{\mathbf{k}}_\lambda \\ 0 \end{bmatrix}\right\rangle\right| \\ &\leqslant \mathbf{ber}\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) \qquad \text{(by the definition of } \mathbf{ber}) \\ &\leqslant \mathbf{ber}\left(\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &\leqslant \mathbf{ber}(Y) \qquad \text{(by inequality (2.1))}. \end{aligned}$$

Hence

$$\mathbf{ber}(X) = \sup_{\lambda \in \Omega}\big|\langle X\hat{\mathbf{k}}_\lambda, \hat{\mathbf{k}}_\lambda\rangle\big| \leqslant \mathbf{ber}(Y).$$

□

Corollary 2.11. *Let $A_i, B_i, X_i \in \mathbb{B}(\mathcal{H}(\Omega))$, $1 \leqslant i \leqslant n$, $r \geqslant 1$ and let f, g be nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$, $t \in [0,\infty)$. Then*

$$\mathbf{ber}^r\Big(\sum_{i=1}^n A_i^* X_i B_i\Big) \leqslant \mathbf{ber}\bigg(\frac{1}{p}\Big[\sum_{i=1}^n B_i^* f^2(|X_i|)B_i\Big]^{rp/2} + \frac{1}{q}\Big[\sum_{i=1}^n A_i^* g^2(|X_i^*|)A_i\Big]^{rq/2}\bigg),$$

where $1/p + 1/q = 1$ and $pr \geqslant qr \geqslant 2$.

In particular, if $\sum\limits_{i=1}^n A_i^* A_i \leqslant I$ and $\sum\limits_{i=1}^n B_i^* B_i \leqslant I$, then

$$\mathbf{ber}^r\Big(\sum_{i=1}^n A_i^* X_i B_i\Big) \leqslant \mathbf{ber}\Big(\frac{1}{p}\sum_{i=1}^n B_i^* f^{rp}(|X_i|)B_i + \frac{1}{q}\sum_{i=1}^n A_i^* g^{rq}(|X_i^*|)A_i\Big).$$

P r o o f. If we replace A, B and X in Theorem 2.7 by operator matrices

$$\begin{bmatrix} A_1 & 0 & \dots & 0 \\ A_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & 0 & \dots & 0 \end{bmatrix}, \quad \begin{bmatrix} B_1 & 0 & \dots & 0 \\ B_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_n & 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_n \end{bmatrix},$$

respectively, then we get

$$\begin{aligned} &\mathbf{ber}^r\left(\begin{bmatrix} \sum\limits_{i=1}^n A_i^* X_i B_i & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &\quad \leqslant \mathbf{ber}\left(\begin{bmatrix} \frac{1}{p}\Big[\sum\limits_{i=1}^n B_i^* f^2(|X_i|)B_i\Big]^{rp/2} + \frac{1}{q}\Big[\sum\limits_{i=1}^n A_i^* g^2(|X_i^*|)A_i\Big]^{rq/2} & 0 \\ 0 & 0 \end{bmatrix}\right). \end{aligned}$$

Now, using Lemma 2.10 we have

$$\begin{aligned} &\mathbf{ber}^r\Big(\sum_{i=1}^n A_i^* X_i B_i\Big) \\ &\quad \leqslant \mathbf{ber}\bigg(\frac{1}{p}\Big[\sum_{i=1}^n B_i^* f^2(|X_i|)B_i\Big]^{rp/2} + \frac{1}{q}\Big[\sum_{i=1}^n A_i^* g^2(|X_i^*|)A_i\Big]^{rq/2}\bigg), \end{aligned}$$

the first inequality. The second inequality follows from inequality (2.2) and this completes the proof. □

In the next theorem we present an inequality involving the generalized Euclidean Berezin number for off-diagonal operator matrices.

Theorem 2.12. *Let* $T_i = \begin{bmatrix} 0 & X_i \\ Y_i & 0 \end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2))$, $1 \leqslant i \leqslant n$. *Then*

$$\begin{aligned}&\mathbf{ber}_p^p(T_1, T_2, \ldots, T_n)\\ &\qquad\leqslant 2^{p-2} \sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|))\, \mathbf{ber}^{1/2}(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|))\end{aligned}$$

for $p \geqslant 1$.

Proof. For every $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}(\Omega_1) \oplus \mathcal{H}(\Omega_2)$ (i.e., $\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2 = 1$). Then

$$\begin{aligned}
\sum_{i=1}^n |\langle T_i \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}, \hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}\rangle|^p &= \sum_{i=1}^n |\langle X_i k_{\lambda_2}, k_{\lambda_1}\rangle + \langle Y_i k_{\lambda_1}, k_{\lambda_2}\rangle|^p \\
&\leqslant \sum_{i=1}^n (|\langle X_i k_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle Y_i k_{\lambda_1}, k_{\lambda_2}\rangle|)^p \qquad \text{(by the triangular inequality)} \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^n |\langle X_i k_{\lambda_2}, k_{\lambda_1}\rangle|^p + |\langle Y_i k_{\lambda_1}, k_{\lambda_2}\rangle|^p \qquad \text{(by the convexity } f(t) = t^p) \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^n \langle f^2(|X_i|) k_{\lambda_2}, k_{\lambda_2}\rangle^{p/2} \langle g^2(|X_i^*|) k_{\lambda_1}, k_{\lambda_1}\rangle^{p/2} + \langle f^2(|Y_i|) k_{\lambda_1}, k_{\lambda_1}\rangle^{p/2} \\
&\quad \times \langle g^2(|Y_i^*|) k_{\lambda_2}, k_{\lambda_2}\rangle^{p/2} \qquad \text{(by Lemma 2.4)} \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^n \langle f^{2p}(|X_i|) k_{\lambda_2}, k_{\lambda_2}\rangle^{1/2} \langle g^{2p}(|X_i^*|) k_{\lambda_1}, k_{\lambda_1}\rangle^{1/2} + \langle f^{2p}(|Y_i|) k_{\lambda_1}, k_{\lambda_1}\rangle^{1/2} \\
&\quad \times \langle g^{2p}(|Y_i^*|) k_{\lambda_2}, k_{\lambda_2}\rangle^{1/2} \qquad \text{(by Lemma 2.3)} \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^n (\langle f^{2p}(|X_i|) k_{\lambda_2}, k_{\lambda_2}\rangle + \langle g^{2p}(|Y_i^*|) k_{\lambda_2}, k_{\lambda_2}\rangle)^{1/2} (\langle f^{2p}(|Y_i|) k_{\lambda_1}, k_{\lambda_1}\rangle \\
&\quad + \langle g^{2p}(|X_i^*|) k_{\lambda_1}, k_{\lambda_1}\rangle)^{1/2} \qquad \text{(by the Cauchy-Schwarz inequality)} \\
&\leqslant \frac{2^p}{2} \sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|))\, \mathbf{ber}^{1/2}(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)) \|k_{\lambda_1}\| \|k_{\lambda_2}\| \\
&= \frac{2^p}{2} \sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|))\, \mathbf{ber}^{1/2}(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)) \\
&\quad \times \Big(\frac{\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2}{2}\Big) \\
&= \frac{2^p}{4} \sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|) + g^{2p}(|Y_i^*|))\, \mathbf{ber}^{1/2}(f^{2p}(|Y_i|) + g^{2p}(|X_i^*|)).
\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{ber}_p^p(T_1,T_2,\ldots,T_n) &= \sup_{(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2}\sum_{i=1}^n |\langle T_i\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)},\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}\rangle|^p\\ &\leqslant 2^{p-2}\sum_{i=1}^n \mathbf{ber}^{1/2}(f^{2p}(|X_i|)+g^{2p}(|Y_i^*|))\,\mathbf{ber}^{1/2}(f^{2p}(|Y_i|)+g^{2p}(|X_i^*|))\end{aligned}$$

as required. $\square$

Theorem 2.13. *Let* $T_i = \begin{bmatrix} A_i & B_i\\ C_i & D_i\end{bmatrix} \in \mathbb{B}(\mathcal{H}(\Omega_1)\oplus\mathcal{H}(\Omega_2))$, $1\leqslant i\leqslant n$ *and* $p\geqslant 1$. *Then*

$$\begin{aligned}&\mathbf{ber}_p^p(T_1,\ldots,T_n)\\ &\quad\leqslant 2^{-p}\sum_{i=1}^n\Big(\mathbf{ber}(A_i)+\mathbf{ber}(D_i)+\sqrt{(\mathbf{ber}(A_i)-\mathbf{ber}(D_i))^2+(\|B_i\|+\|C_i\|)^2}\Big)^p.\end{aligned}$$

Proof. For every $(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2$, let $\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1}\\ k_{\lambda_2}\end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}(\Omega_1)\oplus\mathcal{H}(\Omega_2)$. It follows from

$$\begin{aligned}|\langle T_i\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)},\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}\rangle| &= \left|\left\langle \begin{bmatrix} A_i & B_i\\ C_i & D_i\end{bmatrix}\begin{bmatrix} k_{\lambda_1}\\ k_{\lambda_2}\end{bmatrix}, \begin{bmatrix} k_{\lambda_1}\\ k_{\lambda_2}\end{bmatrix}\right\rangle\right|\\ &= \left|\left\langle \begin{bmatrix} A_ik_{\lambda_1}+B_ik_{\lambda_2}\\ C_ik_{\lambda_1}+D_ik_{\lambda_2}\end{bmatrix}, \begin{bmatrix} k_{\lambda_1}\\ k_{\lambda_2}\end{bmatrix}\right\rangle\right|\\ &= |\langle A_ik_{\lambda_1},k_{\lambda_1}\rangle+\langle B_ik_{\lambda_2},k_{\lambda_1}\rangle+\langle C_ik_{\lambda_1},k_{\lambda_2}\rangle+\langle D_ik_{\lambda_2},k_{\lambda_2}\rangle|\\ &\leqslant |\langle A_ik_{\lambda_1},k_{\lambda_1}\rangle|+|\langle B_ik_{\lambda_2},k_{\lambda_1}\rangle|+|\langle C_ik_{\lambda_1},k_{\lambda_2}\rangle|+|\langle D_ik_{\lambda_2},k_{\lambda_2}\rangle|\end{aligned}$$

that

$$\begin{aligned}\mathbf{ber}_p^p(T_1,\ldots,T_n) &= \sup_{(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2}\sum_{i=1}^n|\langle T_i\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)},\hat{\mathbf{k}}_{(\lambda_1,\lambda_2)}\rangle|^p\\ &\leqslant \sup_{(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2}\sum_{i=1}^n(|\langle A_ik_{\lambda_1},k_{\lambda_1}\rangle|+|\langle B_ik_{\lambda_2},k_{\lambda_1}\rangle|\\ &\qquad\qquad+|\langle C_ik_{\lambda_1},k_{\lambda_2}\rangle|+|\langle D_ik_{\lambda_2},k_{\lambda_2}\rangle|)^p\\ &\leqslant \sum_{i=1}^n\Big(\sup_{(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2}(|\langle A_ik_{\lambda_1},k_{\lambda_1}\rangle|+|\langle B_ik_{\lambda_2},k_{\lambda_1}\rangle|\\ &\qquad\qquad+|\langle C_ik_{\lambda_1},k_{\lambda_2}\rangle|+|\langle D_ik_{\lambda_2},k_{\lambda_2}\rangle|)\Big)^p\\ &\leqslant \sum_{i=1}^n\Big(\sup_{(\lambda_1,\lambda_2)\in\Omega_1\times\Omega_2}(\mathbf{ber}(A_i)\|k_{\lambda_1}\|^2+\mathbf{ber}(D_i)\|k_{\lambda_2}\|^2\\ &\qquad\qquad+(\|B_i\|+\|C_i\|)\|k_{\lambda_1}\|\|k_{\lambda_2}\|)\Big)^p\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{n}\Big(\sup_{\theta\in[0,2\pi]}(\mathbf{ber}(A_i)\cos^2\theta+\mathbf{ber}(D_i)\sin^2\theta \\
&\qquad\qquad + (\|B_i\|+\|C_i\|)\cos\theta\sin\theta)\Big)^p \\
&= 2^{-p}\sum_{i=1}^{n}\big(\mathbf{ber}(A_i)+\mathbf{ber}(D_i) \\
&\qquad\qquad + \sqrt{(\mathbf{ber}(A_i)-\mathbf{ber}(D_i))^2+(\|B_i\|+\|C_i\|)^2}\,\big)^p.
\end{aligned}
$$

This completes the proof. □

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