Stress and Strain Measures

Goals – Stress & Strain Measures

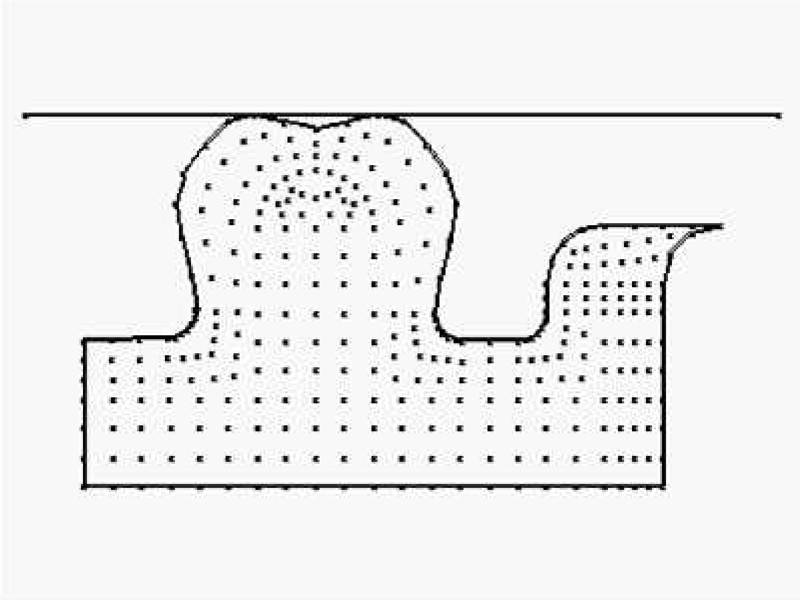
- Definition of a nonlinear elastic problem
- Understand the deformation gradient?
- What are Lagrangian and Eulerian strains?
- What is polar decomposition and how to do it?
- How to express the deformation of an area and volume
- What are Piola-Kirchhoff and Cauchy stresses?

What Is a Nonlinear Elastic Problem?

- **Elastic** (same for linear and nonlinear problems)
  - Stress-strain relation is elastic
  - Deformation disappears when the applied load is removed
  - Deformation is history-independent
  - Potential energy exists (function of deformation)
- **Nonlinear**
  - Stress-strain relation is nonlinear
  - Deformation is large
- **Lagrangian or Material Stress/Strain**:

  when the reference frame is

  undeformed configuration



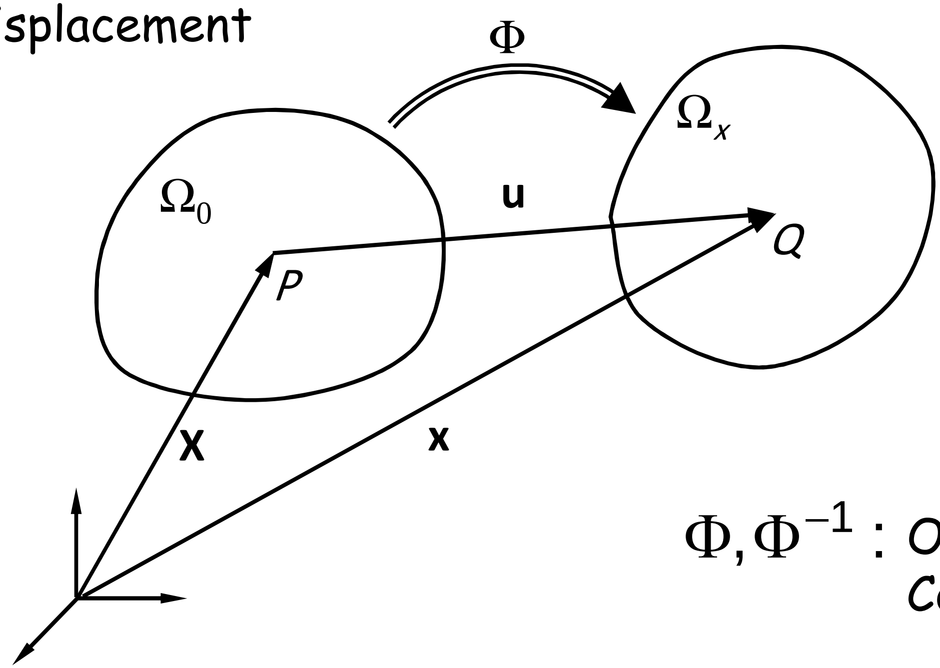
- **Eulerian or Spatial Stress/Strain**:

  when the reference frame is deformed configuration

Deformation and Mapping

- Initial domain Ω_0 is deformed to Ω_x
  - We can think of this as a **mapping** from Ω_0 to Ω_x
- **X**: material point in Ω_0 **x**: material point in Ω_x
- Material point P in Ω_0 is deformed to Q in Ω_x

$$\mathbf{x} = \mathbf{X} + \underset{\text{displacement}}{\mathbf{u}} \quad \Longrightarrow \quad \boxed{\mathbf{x} = \Phi(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)}$$

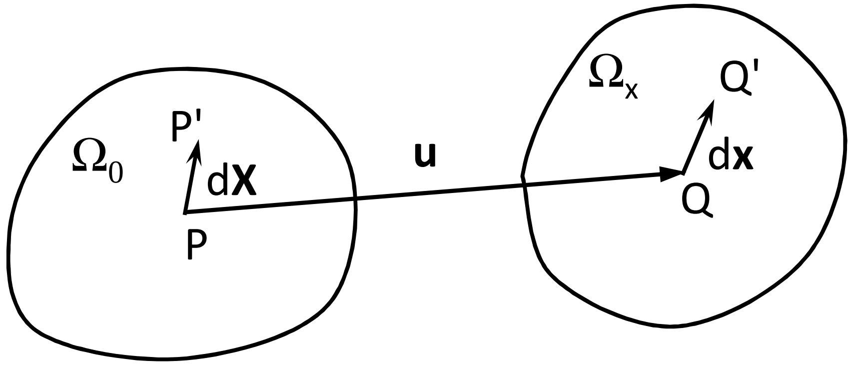


Φ, Φ^{-1} : One-to-one mapping
Continuously differentiable

Deformation Gradient

- Infinitesimal length d**X** in Ω_0 deforms to d**x** in Ω_x
- Remember that the mapping is **continuously differentiable**

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} d\mathbf{X} \quad \Rightarrow \quad d\mathbf{x} = \mathbf{F} d\mathbf{X}$$



- **Deformation gradient**:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \qquad \mathbf{F} = \mathbf{1} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{1} + \nabla_0 \mathbf{u}$$

$$\mathbf{1} = [\delta_{ij}],$$

$$\nabla_0 = \frac{\partial}{\partial \mathbf{X}}, \quad \nabla_x = \frac{\partial}{\partial \mathbf{x}}$$

  - gradient of mapping Φ
  - **Second-order tensor**, **Depend on both** W_0 **and** W_x
  - Due to one-to-one mapping: $\boxed{\det \mathbf{F} \equiv J > 0.}$ $\qquad d\mathbf{X} = \mathbf{F}^{-1} d\mathbf{x}$
  - **F includes both deformation and rigid-body rotation**

Example – Uniform Extension

- Uniform extension of a cube in all three directions

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3$$

- **Continuity requirement**: $\lambda_i > 0$
- Deformation gradient:

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = \lambda_3$: uniform expansion (**dilatation**) or contraction
- Volume change
  - Initial volume: $dV_0 = dX_1 dX_2 dX_3$
  - Deformed volume:

$$dV_x = dx_1 dx_2 dx_3 = \lambda_1 \lambda_2 \lambda_3 dX_1 dX_2 dX_3 = \lambda_1 \lambda_2 \lambda_3 dV_0$$

Green-Lagrange Strain

- Why different strains?
- Length change: $\|d\mathbf{x}\|^2 - \|d\mathbf{X}\|^2 = d\mathbf{x}^T d\mathbf{x} - d\mathbf{X}^T d\mathbf{X}$

$$= d\mathbf{X}^T \mathbf{F}^T \mathbf{F} d\mathbf{X} - d\mathbf{X}^T d\mathbf{X}$$

$$= d\mathbf{X}^T \underbrace{(\mathbf{F}^T \mathbf{F} - \mathbf{1})}_{\text{Ratio of length change}} d\mathbf{X}$$

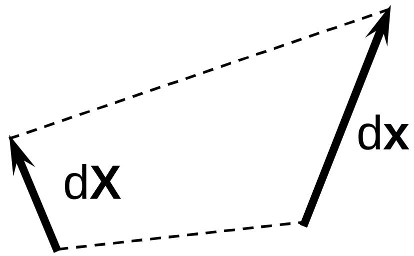
- **Right Cauchy-Green Deformation Tensor**

$$\boxed{\mathbf{C} = \mathbf{F}^T \mathbf{F}}$$

- **Green-Lagrange Strain Tensor**

$$\boxed{\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1})}$$

To match with infinitesimal strain



The effect of rotation is eliminated

Green-Lagrange Strain cont.

- Properties:
  - $\mathbf{E}$ is **symmetric**: $\mathbf{E}^T = \mathbf{E}$
  - No deformation: $\mathbf{F} = \mathbf{1}$, $\mathbf{E} = \mathbf{0}$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

$$\boxed{\begin{aligned}\mathbf{E} &= \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}^T + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}^T \frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right) \\ &= \tfrac{1}{2}\left(\nabla_0\mathbf{u} + \nabla_0\mathbf{u}^T + \nabla_0\mathbf{u}^T\nabla_0\mathbf{u}\right)\end{aligned}}$$

Displacement gradient

Higher-order term

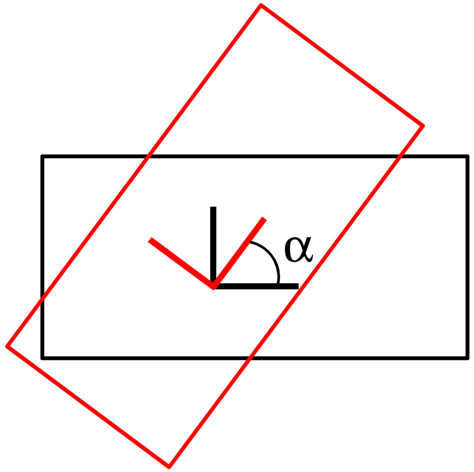
  - When $|\nabla_0 \mathbf{u}| << 1$, $\quad \mathbf{E} \approx \frac{1}{2}\left(\nabla_0\mathbf{u} + \nabla_0\mathbf{u}^T\right) = \varepsilon$

  - **E = 0 for a rigid-body motion**, but $\varepsilon \neq 0$

Example – Rigid-Body Rotation

- Rigid-body rotation

$$x_1 = X_1 \cos\alpha - X_2 \sin\alpha$$
$$x_2 = X_1 \sin\alpha + X_2 \cos\alpha$$
$$x_3 = X_3$$



- Approach 1: using deformation gradient

$$\mathbf{F} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \mathbf{0}$$

Green-Lagrange strain removes rigid-body rotation from deformation

Example – Rigid-Body Rotation cont.

- Approach 2: using displacement gradient

$$u_1 = x_1 - X_1 = X_1(\cos\alpha - 1) - X_2 \sin\alpha$$
$$u_2 = x_2 - X_2 = X_1 \sin\alpha + X_2(\cos\alpha - 1)$$
$$u_3 = x_3 - X_3 = 0$$

$$\nabla_0 \mathbf{u} = \begin{bmatrix} \cos\alpha - 1 & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\nabla_0 \mathbf{u}^T \nabla_0 \mathbf{u} = \begin{bmatrix} 2(1-\cos\alpha) & 0 & 0 \\ 0 & 2(1-\cos\alpha) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

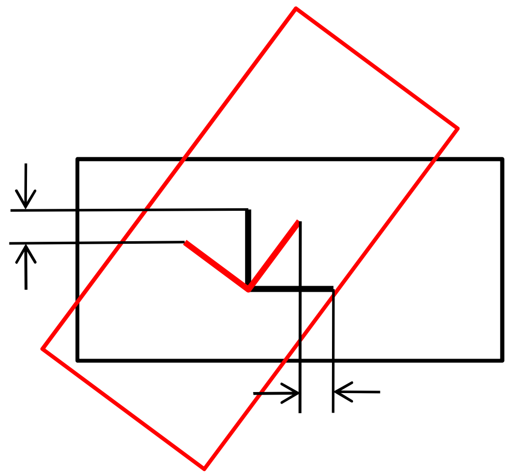
$$\mathbf{E} = \tfrac{1}{2}(\nabla_0 \mathbf{u} + \nabla_0 \mathbf{u}^T + \nabla_0 \mathbf{u}^T \nabla_0 \mathbf{u}) = \mathbf{0}$$

Example – Rigid-Body Rotation cont.

- What happens to engineering strain?

$$u_1 = x_1 - X_1 = X_1(\cos\alpha - 1) - X_2 \sin\alpha$$
$$u_2 = x_2 - X_2 = X_1 \sin\alpha + X_2(\cos\alpha - 1)$$
$$u_3 = x_3 - X_3 = 0$$

$$\pounds = \begin{bmatrix} \cos\alpha - 1 & 0 & 0 \\ 0 & \cos\alpha - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Engineering strain is unable to take care of rigid-body rotation

Eulerian (Almansi) Strain Tensor

- Length change: $\|d\mathbf{x}\|^2 - \|d\mathbf{X}\|^2 = d\mathbf{x}^T d\mathbf{x} - d\mathbf{X}^T d\mathbf{X}$

$$= d\mathbf{x}^T d\mathbf{x} - d\mathbf{x}^T \mathbf{F}^{-T}\mathbf{F}^{-1} d\mathbf{x}$$

$$= d\mathbf{x}^T (\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}) d\mathbf{x}$$

$$= d\mathbf{x}^T (\mathbf{1} - \mathbf{b}^{-1}) d\mathbf{x}$$

- **Left Cauchy-Green Deformation Tensor**

$$\boxed{\mathbf{b} = \mathbf{F}\mathbf{F}^T}$$

$\mathbf{b}^{-1}$: Finger tensor

- **Eulerian (Almansi) Strain Tensor**

$$\boxed{\mathbf{e} = \frac{1}{2}(\mathbf{1} - \mathbf{b}^{-1})}$$

Reference is deformed (current) configuration

Eulerian Strain Tensor cont.

- Properties
  - Symmetric
  - Approach engineering strain when $\frac{\partial \mathbf{u}}{\partial \mathbf{x}} << 1$
  - In terms of displacement gradient

$$\mathbf{e} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}^T}{\partial \mathbf{x}} - \frac{\partial \mathbf{u}^T}{\partial \mathbf{x}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)$$

$$= \frac{1}{2}\left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T - \nabla_x \mathbf{u}^T \nabla_x \mathbf{u} \right)$$

$$\nabla_x = \frac{\partial}{\partial \mathbf{x}}$$

Spatial gradient

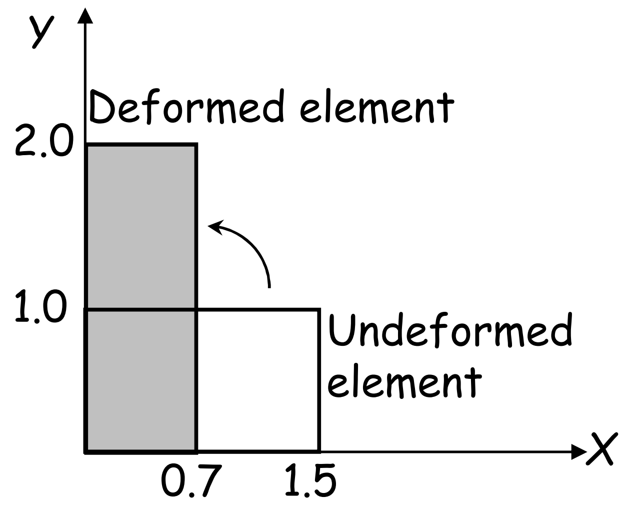
- Relation between **E** and ***e***

$$\mathbf{E} = \mathbf{F}^T \mathbf{e} \mathbf{F}$$

Example – Lagrangian Strain

- Calculate **F** and **E** for deformation in the figure
- Mapping relation in Ω_0

$$\begin{cases} X = \dfrac{3}{4}(s+1) \\[2ex] Y = \dfrac{1}{2}(t+1) \end{cases}$$



- Mapping relation in Ω_x

$$\begin{cases} x(s,t) = 0.35(1-t) \\[2ex] y(s,t) = s+1 \end{cases}$$

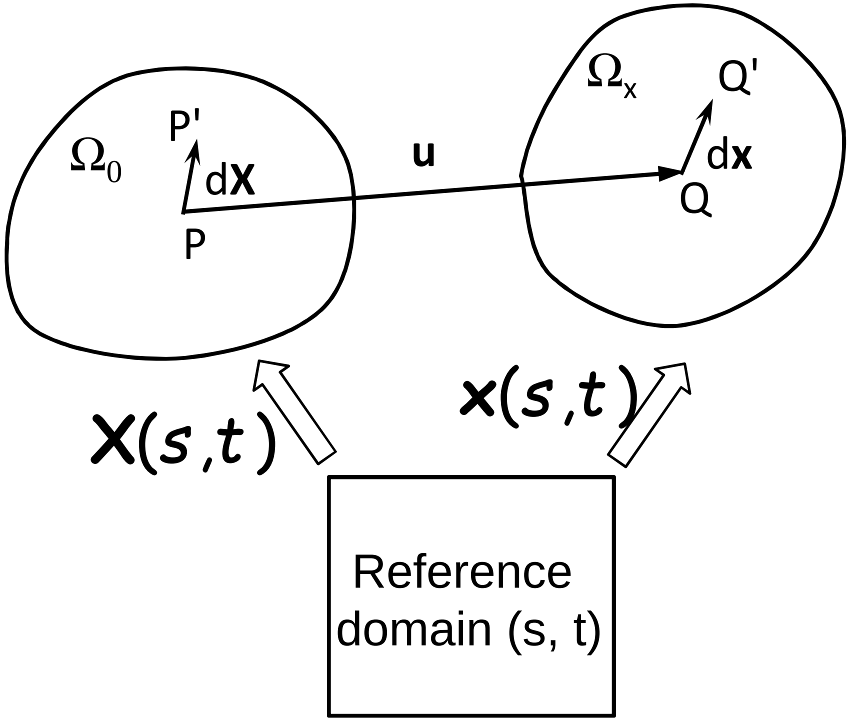
Example – Lagrangian Strain cont.

- Deformation gradient

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \mathbf{s}}\frac{\partial \mathbf{s}}{\partial \mathbf{X}}$$

$$= \begin{bmatrix} 0 & -.35 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4/3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.7 \\ 4/3 & 0 \end{bmatrix}$$



- Green-Lagrange Strain

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \begin{bmatrix} 0.389 & 0 \\ 0 & -0.255 \end{bmatrix}$$

Example – Lagrangian Strain cont.

- Almansi Strain

$$\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^T = \begin{bmatrix} 0.49 & 0 \\ 0 & 1.78 \end{bmatrix}$$

$$\mathbf{e} = \tfrac{1}{2}\left(\mathbf{1} - \mathbf{b}^{-1}\right) = \begin{bmatrix} -0.52 & 0 \\ 0 & 0.22 \end{bmatrix}$$

- Engineering Strain

$$\nabla_0 \mathbf{u} = \mathbf{F} - \mathbf{1} = \begin{bmatrix} -1 & -0.7 \\ 1.33 & -1 \end{bmatrix}$$

$$£ = \tfrac{1}{2}\left(\nabla_0 \mathbf{u} + \nabla_0 \mathbf{u}^T\right) = \begin{bmatrix} -1 & 0.32 \\ 0.32 & -1 \end{bmatrix}$$

Which strain is consistent with actual deformation?

Example – Uniaxial Tension

- Uniaxial tension of incompressible material ($\lambda_1 = \lambda > 1$)
- From incompressibility

$$x_1 = \lambda_1 X_1$$

$$x_2 = \lambda_2 X_2$$

$$x_3 = \lambda_3 X_3$$

$$\lambda_1 \lambda_2 \lambda_3 = 1 \quad \Rightarrow \quad \lambda_2 = \lambda_3 = \lambda^{-1/2}$$

- Deformation gradient and deformation tensor

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

- G-L Strain

$$\mathbf{E} = \frac{1}{2} \begin{bmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & \lambda^{-1} - 1 & 0 \\ 0 & 0 & \lambda^{-1} - 1 \end{bmatrix}$$

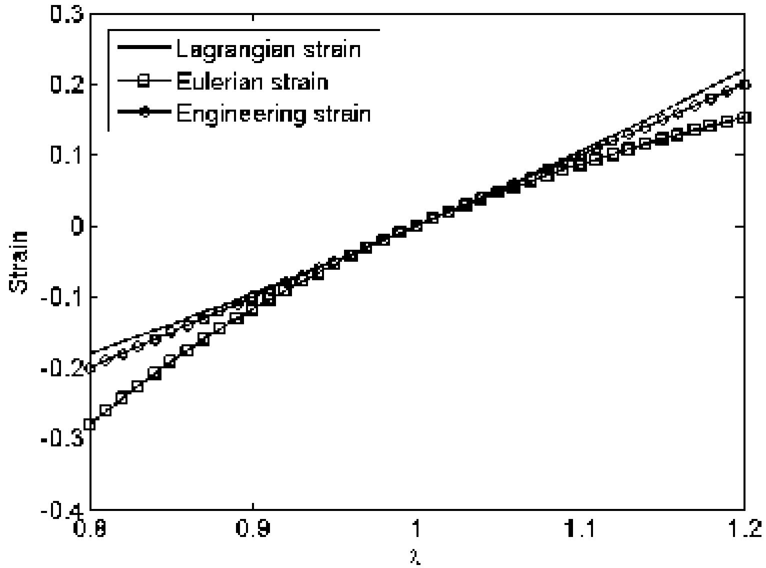
Example – Uniaxial Tension

- Almansi Strain (**b** = **C**)

$$\mathbf{b}^{-1} = \begin{bmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \qquad \mathbf{e} = \frac{1}{2}\begin{bmatrix} 1-\lambda^{-2} & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

- Engineering Strain

$$£ = \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda^{-1/2} - 1 & 0 \\ 0 & 0 & \lambda^{-1/2} - 1 \end{bmatrix}$$



- Difference

$$E_{11} = \frac{1}{2}(\lambda^2 - 1) \quad e_{11} = \frac{1}{2}(1-\lambda^{-2}) \quad \varepsilon_{11} = \lambda - 1$$