

9-12-2025

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Recommended Citation

BAKHERAD, M, YAMANCI, U, & ALOMARI, M (2025). Improved Berezin radius inequalities for functional Hilbert space operators. *Turkish Journal of Mathematics* 49 (5): 545-561. <https://doi.org/10.55730/1300-0098.3607>



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Improved Berezin radius inequalities for functional Hilbert space operators

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Received: 22.03.2025

Accepted/Published Online: 16.06.2025

Final Version: 12.09.2025

Abstract: The Berezin number of an operator S is defined by $\mathbf{ber}(S) = \sup_{\lambda \in \mathbb{R}} |\tilde{S}(\lambda)| = \sup_{\lambda \in \mathbb{R}} |\langle S\hat{k}_\lambda, \hat{k}_\lambda \rangle|$, where $\tilde{S}(\lambda) = \langle S\hat{k}_\lambda, \hat{k}_\lambda \rangle$ ($\lambda \in \Theta$) is the Berezin transform of an operator S on the functional Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$ and $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ is the normalized reproducing kernel of $\mathcal{H}$. In this paper, we show an improvement of the classical Cauchy–Schwarz inequality involving the Berezin number. Moreover, some Berezin numbers inequalities are proven for invertible operators.

Key words: Berezin number, berezin symbol, furuta’s inequality

1. Introduction

A Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$ consisting of complex-valued functions defined on a nonempty open set $\Theta \subset \mathbb{C}$ which has the property that point evaluations are continuous, is called a *functional Hilbert space*. The point evaluations are continuous means for each $\lambda \in \Theta$, the map $f \mapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. For each $\lambda \in \Theta$, there is a unique element k_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$ by Riesz representation theorem. The collection $\{k_\lambda : \lambda \in \Theta\}$ is known as the *reproducing kernel* of $\mathcal{H}$. The reproducing kernel of a functional Hilbert space $\mathcal{H}$ with $\{e_n\}$ as an orthonormal basis is $k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z)$. Let $\hat{k}_\lambda = k_\lambda / \|k_\lambda\|$ be the normalized reproducing kernel of $\mathcal{H}$, where $\lambda \in \Theta$. The function $\tilde{S}$ defined on Θ by $\tilde{S}(\lambda) = \langle S\hat{k}_\lambda, \hat{k}_\lambda \rangle$ is the *Berezin symbol* of a bounded linear operator S on $\mathcal{H}$. The *Berezin set* and the *Berezin number* of the operator S are defined by

$$\mathbf{Ber}(S) = \{\tilde{S}(\lambda) : \lambda \in \Theta\} \text{ and } \mathbf{ber}(S) = \sup\{|\tilde{S}(\lambda)| : \lambda \in \Theta\},$$

respectively. These definitions are named in honor of Felix Berezin, who introduced these notions in [2]. For our purpose, we set the *Berezin norm* of an operator as $\|S\|_{\mathbf{ber}} = \sup\{|\langle S\hat{k}_{\lambda_1}, \hat{k}_{\lambda_2} \rangle| : \lambda_1, \lambda_2 \in \Theta\}$. Clearly, the Berezin symbol $\tilde{S}$ is a bounded function on Θ whose values lie in the numerical range of the operator S , and hence

$$\mathbf{Ber}(S) \subseteq W(S) \text{ and } \mathbf{ber}(S) \leq w(S),$$

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2010 AMS Mathematics Subject Classification: 47A63

where $W(S) := \{\langle Sf, f \rangle : f \in \mathcal{H} \text{ and } \|f\|_{\mathcal{H}} = 1\}$ is the numerical range of the operator S and

$$w(S) := \sup \{|\langle Sf, f \rangle| : f \in \mathcal{H} \text{ and } \|f\|_{\mathcal{H}} = 1\}$$

is the numerical radius of S . The Berezin number of operators T and S satisfies the following properties:

- (i) $\mathbf{ber}(\beta T) = |\beta| \mathbf{ber}(T)$ for all $\beta \in \mathbb{C}$.
- (ii) $\mathbf{ber}(T + S) \leq \mathbf{ber}(T) + \mathbf{ber}(S)$.

The Berezin symbol has been extensively studied for Toeplitz and Hankel operators in Hardy and Bergman spaces. An important property of the Berezin symbol is presented next. If $\tilde{S}(\lambda) = \tilde{T}(\lambda)$ for all $\lambda \in \Theta$, then $S = T$. Therefore, the Berezin symbol uniquely characterizes the operator. Notably, several significant results concerning the Berezin number were established in [1, 3–6, 14–16, 18, 21, 22] very recently. Among many techniques in obtaining numerical radius and Berezin number inequalities is the study of certain scalar ones.

Let $\mathcal{L}(\mathcal{H}(\Theta))$ denotes the set of all bounded linear operators on a complex Hilbert space $\mathcal{H}(\Theta)$. For $S \in \mathbb{B}(\mathcal{H})$, let $S = \Re(S) + i\Im(S)$ be the Cartesian decomposition of S . We say that an operator S is accretive if $\Re(S)$ is positive and is dissipative if $\Im(S)$ is positive. If both $\Re(S)$ and $\Im(S)$ are positive, S is called accretive-dissipative. For a positive operator $S \in \mathcal{L}(\mathcal{H}(\Theta))$ we have [13]

$$\langle Sa, a \rangle^s \leq \langle S^s a, a \rangle, \quad \text{for } a \in \mathcal{H}, \|a\| = 1 \text{ and } s \geq 1. \quad (1.1)$$

The last inequality is reversed if $s \in [0, 1]$. The well known Schwarz inequality is

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle \quad (u, v \in \mathcal{H}(\Theta)).$$

Let $S \in \mathcal{L}(\mathcal{H}(\Theta))$ be positive and $a, b \in \mathcal{H}(\Theta)$. By taking $u = S^{1/2}a$ and $v = S^{1/2}b$ in last inequality we get the following Schwarz inequality

$$|\langle Sa, b \rangle|^2 \leq \langle Sb, b \rangle \langle Sa, a \rangle. \quad (1.2)$$

Kato in [19] introduced the mixed Schwarz inequality

$$|\langle Sa, b \rangle|^2 \leq \langle |S|^{2s} a, a \rangle \langle |S^{*}|^{2(1-s)} b, b \rangle, \quad 0 \leq s \leq 1, \quad (1.3)$$

where $|S| = (S^*S)^{1/2}$ for $S \in \mathcal{L}(\mathcal{H}(\Theta))$ and a, b are vectors in $\mathcal{H}(\Theta)$. Furuta in [13] assert another extension of the Schwarz inequality as follows

$$\left| \langle S |S|^{s+t-1} a, b \rangle \right|^2 \leq \langle |S|^{2s} a, a \rangle \langle |S|^{2t} b, b \rangle \quad (1.4)$$

for any $a, b \in \mathcal{H}(\Theta)$ and $s, t \in [0, 1]$ with $s + t \geq 1$.

Dragomir [12] provided a generalization of Furuta's inequality

$$|\langle T S R Q a, b \rangle|^2 \leq \langle Q^* |R|^2 Q a, a \rangle \langle T |S^*|^2 T^* b, b \rangle, \quad (1.5)$$

where $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$ and $a, b \in \mathcal{H}(\Theta)$. Moreover, very recently Dragomir [11] proved a reverse of the Schwarz inequality as follows:

Lemma 1.1 *Let $u, v \in \mathcal{H}$ and $\alpha \in \mathbb{C}$. Then*

$$(1) \quad \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 \leq \|u - \alpha v\|^2 \|v\|^2.$$

$$(2) \quad \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 \leq \frac{1}{4} \|u - v\|^2 \|u + v\|^2.$$

Lemma 1.2 [10] *If $u, v \in \mathcal{H}$, and $p \geq 2$, then*

$$\|u\|^p + \|v\|^p \leq \frac{1}{2} (\|u + v\|^p + \|u - v\|^p).$$

Lemma 1.3 [8] *Let $g : I \rightarrow \mathcal{R}$ be a convex function on the interval I and $a, b \in I$. Then*

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &\leq (1-\alpha)g\left(\frac{(1-\alpha)a + (1+\alpha)b}{2}\right) + \alpha g\left(\frac{(2-\alpha)a + \alpha b}{2}\right) \\ &\leq \int_0^1 g((1-t)a + tb) dt \end{aligned} \quad (1.6)$$

$$\leq \frac{1}{2} (g((1-\alpha)a + \alpha b) + (1-\alpha)g(a) + \alpha g(b)) \quad (1.7)$$

$$\leq \frac{g(a) + g(b)}{2}, \quad (1.8)$$

where $\alpha \in [0, 1]$.

In this paper, we present a refinement of inequality (1.5) and then by using this we show a refinement of the Schwarz-Berezin inequality. Moreover, we prove some Berezin number inequalities involving convex functions.

2. Berezin number inequalities for invertible operators

A basic inequality for the numerical radius is the power inequality

$$w(A^n) \leq w(A)^n, \quad n \geq 1.$$

So, the following questions are natural:

Is it true that the above inequality is also provided for Berezin number of operators? For which operator classes, there exists a number $C > 0$ such that

$$\mathbf{ber}(A)^n \leq C(\mathbf{ber}(A^n)) \quad \text{for all } n?$$

In this section, we aim to address this question and derive upper bounds for the Berezin number of invertible operators

The following lemmas are required to establish our new results

Lemma 2.1 [7] *If $a, b, e \in \mathcal{H}$ and $\|e\| = 1$, then*

$$2|\langle a, e \rangle \langle e, b \rangle| \leq \|a\| \|b\| + |\langle a, b \rangle|.$$

Lemma 2.2 [17] If $T \in \mathcal{L}(\mathcal{H}(\Omega))$, $k \in \mathbb{R}$ and $\eta \in \mathbb{C} - \{0\}$ are such that $\|T - \eta I\|_{\mathbf{ber}} \leq k$, then

$$\|T\|_{\mathbf{ber}}^2 \leq \frac{\mathbf{ber}^2(T)}{1 - \left(\frac{k}{|\eta|}\right)^2},$$

where $k \leq |\eta|$.

Now we give the lower bound for $\mathbf{ber}^2(T)$.

Theorem 2.3 If $T \in \mathcal{L}(\mathcal{H}(\Omega))$, $k \in \mathbb{R}$ and $\eta \in \mathbb{C} - \{0\}$ are such that $\|T - \eta I\|_{\mathbf{ber}} \leq k$ and $k \leq |\eta|$, then

$$\left(2 - \frac{|\eta|^2}{|\eta|^2 - k^2}\right) \mathbf{ber}^2(T) \leq \mathbf{ber}(T^2).$$

Proof Placing $a = T\hat{k}_\mu$, $b = T^*\hat{k}_\mu$ and $e = \hat{k}_\mu$ in Lemma 2.1, we have that

$$2 \left| \tilde{T}(\mu) \right|^2 \leq \left| \tilde{T}^2(\mu) \right| + \left\| T\hat{k}_\mu \right\| \left\| T^*\hat{k}_\mu \right\|$$

and

$$2 \sup_{\mu \in \Theta} \left| \tilde{T}(\mu) \right|^2 \leq \sup_{\mu \in \Theta} \left| \tilde{T}^2(\mu) \right| + \sup_{\mu \in \Theta} \left(\left\| T\hat{k}_\mu \right\| \left\| T^*\hat{k}_\mu \right\| \right),$$

which gives by Lemma 2.2

$$2\mathbf{ber}^2(T) \leq \mathbf{ber}(T^2) + \|T\|_{\mathbf{ber}}^2 \leq \mathbf{ber}(T^2) + \frac{|\eta|^2}{|\eta|^2 - k^2} \mathbf{ber}^2(T).$$

So, we obtain that

$$\left(2 - \frac{|\eta|^2}{|\eta|^2 - k^2}\right) \mathbf{ber}^2(T) \leq \mathbf{ber}(T^2).$$

□

Theorem 2.4 If $T \in \mathcal{L}(\mathcal{H}(\Omega))$ is an invertible accretive-dissipative, $s = \inf_{\mu \in \Theta} \left\| T\hat{k}_\mu \right\| / \|T\|_{\mathbf{ber}}$ and $p \geq 2$, then

$$\frac{(1 + s^p) \|T\|^p}{2^p} \leq \mathbf{ber}^p(T).$$

Proof Putting $a = T\hat{k}_\mu$ and $b = T^*\hat{k}_\mu$ in Lemma 1.2, we have that

$$\left\| T\hat{k}_\mu \right\|^p + \left\| T^*\hat{k}_\mu \right\|^p \leq \frac{1}{2} \left(\left\| T\hat{k}_\mu + T^*\hat{k}_\mu \right\|^p + \left\| T\hat{k}_\mu - T^*\hat{k}_\mu \right\|^p \right)$$

for all $\mu \in \Theta$.

By definition of s ,

$$s^p \left\| T \widehat{k}_\mu \right\|^p + \left\| T^* \widehat{k}_\mu \right\|^p \leq \frac{1}{2} \left(\left\| T \widehat{k}_\mu + T^* \widehat{k}_\mu \right\|^p + \left\| T \widehat{k}_\mu - T^* \widehat{k}_\mu \right\|^p \right)$$

for all $\mu \in \Theta$. Taking the supremum over $\mu \in \Theta$, we obtain that

$$(1 + s^p) \|T\|_{\mathbf{ber}}^p \leq \frac{1}{2} (\|T + T^*\|_{\mathbf{ber}}^p + \|T - T^*\|_{\mathbf{ber}}^p).$$

It follows from $\frac{T^*+T}{2}$ and $\frac{T-T^*}{2i}$ are positive that

$$\begin{aligned} \|T + T^*\|_{\mathbf{ber}}^p + \|T - T^*\|_{\mathbf{ber}}^p &= \mathbf{ber}^p(T + T^*) + \mathbf{ber}^p(T - T^*) \\ &\leq (\mathbf{ber}(T) + \mathbf{ber}(T^*))^p + (\mathbf{ber}(T) + \mathbf{ber}(T^*))^p \\ &= 2^{p+1} \mathbf{ber}^p(T). \end{aligned}$$

Hence,

$$\frac{(1 + s^p) \|T\|^p}{2^p} \leq \mathbf{ber}^p(T).$$

□

Theorem 2.5 *If $T \in \mathcal{L}(\mathcal{H}(\Omega))$ is an invertible accretive-dissipative, $\beta \in \mathbb{C} - \{0\}$ and $s = \inf_{\mu \in \Theta} \left\| T \widehat{k}_\mu \right\| / \|T\|_{\mathbf{ber}}$, then*

$$\mathbf{ber}^2(T) - \mathbf{ber}(T^2) \leq \inf_{\beta} \frac{\|\beta T \pm T^*\|_{\mathbf{ber}}^2}{1 + s^2}.$$

Proof Placing $u = \beta T \widehat{k}_\mu$ and $v = T^* \widehat{k}_\mu$ in Lemma 1.1(1), we obtain that

$$\left\| \beta T \widehat{k}_\mu \right\|^2 \left\| T^* \widehat{k}_\mu \right\|^2 - \left| \left\langle \beta T \widehat{k}_\mu, T^* \widehat{k}_\mu \right\rangle \right|^2 \leq \left\| \alpha \beta T \widehat{k}_\mu \right\|^2 \left\| \beta T \widehat{k}_\mu - T^* \widehat{k}_\mu \right\|^2$$

for all $\mu \in \Theta$. Taking the supremum over $\mu \in \Theta$ gives

$$\sup_{\mu \in \Theta} \left(\left\| T \widehat{k}_\mu \right\| \left\| T^* \widehat{k}_\mu \right\| \right)^2 \leq \mathbf{ber}^2(T^2) + \|T\|_{\mathbf{ber}}^2 \|\alpha \beta T - T^*\|_{\mathbf{ber}}^2. \quad (2.1)$$

On the other hand, by Lemma 2.1 with $a = T \widehat{k}_\mu$, $b = T^* \widehat{k}_\mu$ and $e = \widehat{k}_\mu$, we have that

$$2 \left| \left\langle T \widehat{k}_\mu, \widehat{k}_\mu \right\rangle \right|^2 \leq \left\| T \widehat{k}_\mu \right\| \left\| T^* \widehat{k}_\mu \right\| + \left| \left\langle T^2 \widehat{k}_\mu, \widehat{k}_\mu \right\rangle \right|$$

and

$$2 \sup_{\mu \in \Theta} \left| \left\langle T \widehat{k}_\mu, \widehat{k}_\mu \right\rangle \right|^2 \leq \sup_{\mu \in \Theta} \left(\left\| T \widehat{k}_\mu \right\| \left\| T^* \widehat{k}_\mu \right\| \right) + \sup_{\mu \in \Theta} \left| \left\langle T^2 \widehat{k}_\mu, \widehat{k}_\mu \right\rangle \right|,$$

which gives

$$2 \mathbf{ber}^2(T) - \mathbf{ber}(T^2) \leq \sup_{\mu \in \Theta} \left(\left\| T \widehat{k}_\mu \right\| \left\| T^* \widehat{k}_\mu \right\| \right).$$

Therefore, by (2.1) for $\alpha = 1$,

$$(2\mathbf{ber}^2(T) - \mathbf{ber}(T^2))^2 \leq \mathbf{ber}^2(T^2) + \|T\|_{\mathbf{ber}}^2 \|\beta T - T^*\|_{\mathbf{ber}}^2 \quad (2.2)$$

and applying (2.2) and Theorem 2.4,

$$\mathbf{ber}^2(T) - \mathbf{ber}(T^2) \leq \frac{\|\beta T - T^*\|_{\mathbf{ber}}^2}{1 + s^2},$$

whence

$$\mathbf{ber}^2(T) - \mathbf{ber}(T^2) \leq \inf_{\beta} \frac{\|\beta T - T^*\|_{\mathbf{ber}}^2}{1 + s^2}.$$

Replacing T by iT gives the following inequality

$$\mathbf{ber}^2(T) - \mathbf{ber}(T^2) \leq \inf_{\beta} \frac{\|\beta T + T^*\|_{\mathbf{ber}}^2}{1 + s^2}.$$

□

Using Theorem 2.4, we have the following interest result.

Theorem 2.6 *Let T be an invertible accretive-dissipative and $p = 2$. Then*

$$\|T\|_{\mathbf{ber}}^2 \leq 2\mathbf{ber}^2(T) \text{ or } \|T^{-1}\|_{\mathbf{ber}}^2 \leq 2\mathbf{ber}^2(T^{-1}).$$

Proof For any $\mu \in \Theta$,

$$\frac{1}{\|T^{-1}\|_{\mathbf{ber}}} \leq \|T\hat{k}_{\mu}\|.$$

It follows from $(\|T^{-1}\|_{\mathbf{ber}} \|T\|_{\mathbf{ber}})^{-1} \leq s = \inf_{\mu \in \Theta} \|T\hat{k}_{\mu}\| / \|T\|_{\mathbf{ber}}$, that

$$\|T\|_{\mathbf{ber}}^2 \leq \frac{\|T + T^*\|_{\mathbf{ber}}^2 + \|T - T^*\|_{\mathbf{ber}}^2}{2 \left((\|T^{-1}\|_{\mathbf{ber}} \|T\|_{\mathbf{ber}})^{-2} + 1 \right)},$$

whence

$$\frac{1}{\|T^{-1}\|_{\mathbf{ber}}^2} + \|T\|_{\mathbf{ber}}^2 \leq \frac{1}{2} \left(\|T + T^*\|_{\mathbf{ber}}^2 + \|T - T^*\|_{\mathbf{ber}}^2 \right) \leq 4\mathbf{ber}^2(T).$$

If $\|T^{-1}\|_{\mathbf{ber}} \leq \|T\|_{\mathbf{ber}}$, then

$$\frac{1}{\|T\|_{\mathbf{ber}}^2} + \|T\|_{\mathbf{ber}}^2 \leq 4\mathbf{ber}^2(T).$$

Putting T by $T/\|T\|_{\mathbf{ber}}$ in the last inequality, we have

$$\|T\|_{\mathbf{ber}}^2 \leq 2\mathbf{ber}^2(T).$$

On the other hand, if $\|T\|_{\mathbf{ber}} \leq \|T^{-1}\|_{\mathbf{ber}}$, by replacing T by T^{-1} in above inequality, we have the desired result. □

3. Some results for the Berezin number inequality

In this section, we show several results for the Berezin number inequality. First we need the following lemma:

Lemma 3.1 *Let $R, T \in \mathcal{L}(\mathcal{H}(\Theta))$ be positive operators. Then*

$$(1) \quad \frac{1}{2} \mathbf{ber}^2(R+T) \leq \mathbf{ber}(R^2+T^2).$$

$$(2) \quad \mathbf{ber}^2(R) \leq \mathbf{ber}(R^2).$$

Proof Let $\widehat{k}_\nu \in \mathcal{H}(\Theta)$. Then

$$\begin{aligned} \langle (R+T)\widehat{k}_\nu, \widehat{k}_\nu \rangle^2 &= (\langle R\widehat{k}_\nu, \widehat{k}_\nu \rangle + \langle T\widehat{k}_\nu, \widehat{k}_\nu \rangle)^2 \\ &= \langle R\widehat{k}_\nu, \widehat{k}_\nu \rangle^2 + \langle T\widehat{k}_\nu, \widehat{k}_\nu \rangle^2 + 2\langle R\widehat{k}_\nu, \widehat{k}_\nu \rangle \langle T\widehat{k}_\nu, \widehat{k}_\nu \rangle \\ &\leq \langle R^2\widehat{k}_\nu, \widehat{k}_\nu \rangle + \langle T^2\widehat{k}_\nu, \widehat{k}_\nu \rangle + 2\langle R\widehat{k}_\nu, \widehat{k}_\nu \rangle \langle T\widehat{k}_\nu, \widehat{k}_\nu \rangle \quad (\text{by (1.1)}) \\ &\leq \langle R^2\widehat{k}_\nu, \widehat{k}_\nu \rangle + \langle T^2\widehat{k}_\nu, \widehat{k}_\nu \rangle + \langle R\widehat{k}_\nu, \widehat{k}_\nu \rangle^2 + \langle T\widehat{k}_\nu, \widehat{k}_\nu \rangle^2 \\ &\quad (\text{by the AM-GM inequality}) \\ &\leq \langle R^2\widehat{k}_\nu, \widehat{k}_\nu \rangle + \langle T^2\widehat{k}_\nu, \widehat{k}_\nu \rangle + \langle R^2\widehat{k}_\nu, \widehat{k}_\nu \rangle + \langle T^2\widehat{k}_\nu, \widehat{k}_\nu \rangle \quad (\text{by (1.1)}) \\ &= 2\langle R^2\widehat{k}_\nu, \widehat{k}_\nu \rangle + 2\langle T^2\widehat{k}_\nu, \widehat{k}_\nu \rangle \\ &= 2\langle (R^2+T^2)\widehat{k}_\nu, \widehat{k}_\nu \rangle. \end{aligned}$$

Taking the supremum over all unit vectors $\widehat{k}_\nu \in \mathcal{H}(\Theta)$ we get the first result. If we use (1.1) and then take the supremum over all unit vectors $\widehat{k}_\nu \in \mathcal{H}(\Theta)$ we get the second result. $\square$

Remark 3.2 *Suppose $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$ and $p \geq 1$. If we use inequality (1.5), the AM-GM inequality and inequality (1.1), respectively, we have*

$$\begin{aligned} \left| \langle T S R Q \widehat{k}_\nu, \widehat{k}_\nu \rangle \right|^p &\leq \left\langle Q^* |R|^2 Q \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{p}{2}} \left\langle T |S^*|^2 T^* \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{p}{2}} \\ &\leq \frac{\left\langle Q^* |R|^2 Q \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^p + \left\langle T |S^*|^2 T^* \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^p}{2} \\ &\leq \frac{\left\langle (Q^* |R|^2 Q)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle + \left\langle (T |S^*|^2 T^*)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle}{2} \\ &= \frac{1}{2} \left\langle (Q^* |R|^2 Q)^p + (T |S^*|^2 T^*)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \end{aligned}$$

for all unit vectors $\widehat{k}_\nu \in \mathcal{H}(\Theta)$. Now, we take the supremum over all unit vectors $\widehat{k}_\nu \in \mathcal{H}(\Theta)$ we get

$$\mathbf{ber}^p(TSRQ) \leq \frac{1}{2} \mathbf{ber}((Q^* |R|^2 Q)^p + (T |S^*|^2 T^*)^p), \quad (3.1)$$

where $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$ and $p \geq 1$.

Remark 3.3 Assume $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$ and $p \geq 1$. Then applying (3.1) we have

$$\begin{aligned} \mathbf{ber}^p(TSRQ) &\leq \frac{1}{2} \mathbf{ber}((Q^* |R|^2 Q)^p + (T |S^*|^2 T^*)^p) \\ &\leq \frac{1}{2} \left(\mathbf{ber}((Q^* |R|^2 Q)^p) + \mathbf{ber}((T |S^*|^2 T^*)^p) \right). \end{aligned}$$

If we replace Q by $\sqrt{t}Q$ and T by $\frac{T}{\sqrt{t}}$ ($t > 0$) and take minimum on t ($t > 0$) we get the Schwarz-Berezin inequality

$$\begin{aligned} \mathbf{ber}^p(TSRQ) &\leq \frac{1}{2} \min_{t>0} \left(t^p \mathbf{ber}((Q^* |R|^2 Q)^p) + \frac{1}{t^p} \mathbf{ber}((T |S^*|^2 T^*)^p) \right) \\ &= \mathbf{ber}((Q^* |R|^2 Q)^p) \mathbf{ber}((T |S^*|^2 T^*)^p). \end{aligned} \quad (3.2)$$

In the following theorem, we deduce a refinement of inequality (3.1).

Theorem 3.4 Let $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$, $\lambda \in [0, 1]$ and $p \geq 1$. Then

$$\begin{aligned} \mathbf{ber}^{2p}(TSRQ) &\leq \frac{1}{2} \lambda \left(\mathbf{ber}((Q^* |R|^2 Q)^{2p} + (T |S^*|^2 T^*)^{2p}) \right) \\ &\quad + \frac{1}{2} (1 - \lambda) \mathbf{ber}^p(TSRQ) \mathbf{ber}((Q^* |R|^2 Q)^p + (T |S^*|^2 T^*)^p) \\ &\leq \frac{1}{2} \mathbf{ber}((Q^* |R|^2 Q)^{2p} + (T |S^*|^2 T^*)^{2p}). \end{aligned} \quad (3.3)$$

Proof Let $\widehat{k}_\nu, \widehat{k}_\mu \in \mathcal{H}(\Theta)$. Applying inequalities (1.5) and (1.1) we have

$$\begin{aligned} &\left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\mu \right\rangle \right|^{2p} \\ &= \lambda \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\mu \right\rangle \right|^{2p} + (1 - \lambda) \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\mu \right\rangle \right|^p \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\mu \right\rangle \right|^p \\ &\leq \lambda \left\langle Q^* |R|^2 Q \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^p \left\langle T |S^*|^2 T^* \widehat{k}_\mu, \widehat{k}_\mu \right\rangle^p \\ &\quad \text{(by (1.5))} \\ &\quad + (1 - \lambda) \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\mu \right\rangle \right|^p \left\langle Q^* |R|^2 Q \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{p}{2}} \left\langle T |S^*|^2 T^* \widehat{k}_\mu, \widehat{k}_\mu \right\rangle^{\frac{p}{2}} \\ &\leq \lambda \left\langle (Q^* |R|^2 Q)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \left\langle (T |S^*|^2 T^*)^p \widehat{k}_\mu, \widehat{k}_\mu \right\rangle \\ &\quad + (1 - \lambda) \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\mu \right\rangle \right|^p \sqrt{\left\langle (Q^* |R|^2 Q)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \left\langle (T |S^*|^2 T^*)^p \widehat{k}_\mu, \widehat{k}_\mu \right\rangle} \\ &\quad \text{(by (1.1)).} \end{aligned} \quad (3.4)$$

for all $\lambda \in [0, 1]$ and $p \geq 1$. If we put $\widehat{k}_\nu = \widehat{k}_\mu$ we get

$$\begin{aligned}
& \left| \left\langle (TSRQ) \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right|^{2p} \\
& \leq \lambda \left\langle \left(Q^* |R|^2 Q \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \left\langle \left(T |S^*|^2 T^* \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \\
& \quad + (1 - \lambda) \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right|^p \left\langle \left(Q^* |R|^2 Q \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} \left\langle \left(T^* |S^*|^2 T \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} \\
& \hspace{15em} \text{(by (3.4))} \\
& \leq \lambda \left(\frac{\left\langle \left(Q^* |R|^2 Q \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle + \left\langle \left(T^* |S^*|^2 T \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle}{2} \right)^2 \\
& \quad + \frac{1}{2} (1 - \lambda) \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right|^p \left(\left\langle \left(Q^* |R|^2 Q \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle + \left\langle \left(T |S^*|^2 T^* \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right) \\
& \hspace{15em} \text{(by the AM-GM inequality)} \\
& \leq \lambda \left(\frac{\left\langle \left(Q^* |R|^2 Q \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^2 + \left\langle \left(T |S^*|^2 T^* \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^2}{2} \right) \\
& \quad + \frac{1}{2} (1 - \lambda) \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right|^p \left(\left\langle \left(Q^* |R|^2 Q \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle + \left\langle \left(T |S^*|^2 T^* \right)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right) \\
& \hspace{15em} \text{(by the convexity of } f(t) = t^2 \text{)} \\
& \leq \lambda \left(\frac{\left\langle \left(Q^* |R|^2 Q \right)^{2p} \widehat{k}_\nu, \widehat{k}_\nu \right\rangle + \left\langle \left(T |S^*|^2 T^* \right)^{2p} \widehat{k}_\nu, \widehat{k}_\nu \right\rangle}{2} \right) \\
& \quad + \frac{1}{2} (1 - \lambda) \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right|^p \left\langle \left[\left(Q^* |R|^2 Q \right)^p + \left(T |S^*|^2 T^* \right)^p \right] \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \quad \text{(by (1.1))} \\
& \leq \frac{1}{2} \lambda \left\langle \left[\left(Q^* |R|^2 Q \right)^{2p} + \left(T |S^*|^2 T^* \right)^{2p} \right] \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \\
& \quad + \frac{1}{2} (1 - \lambda) \left| \left\langle TSRQ \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right|^p \left\langle \left[\left(Q^* |R|^2 Q \right)^p + \left(T |S^*|^2 T^* \right)^p \right] \widehat{k}_\nu, \widehat{k}_\nu \right\rangle.
\end{aligned}$$

Taking the supremum over all unit vectors $\widehat{k}_\nu \in \mathcal{H}(\Theta)$ we get the first inequality. To prove the second inequality, we have

$$\begin{aligned}
& \frac{1}{2} \lambda \mathbf{ber}((Q^* |R|^2 Q)^{2p} + (T |S^*|^2 T^*)^{2p}) \\
& \quad + \frac{1}{2} (1 - \lambda) \mathbf{ber}^p(TSRQ) \mathbf{ber}((Q^* |R|^2 Q)^p + (T |S^*|^2 T^*)^p) \\
& \leq \frac{1}{2} \lambda \mathbf{ber}((Q^* |R|^2 Q)^{2p} + (T |S^*|^2 T^*)^{2p}) \\
& \quad + \frac{1}{4} (1 - \lambda) \mathbf{ber}((Q^* |R|^2 Q)^p + (T |S^*|^2 T^*)^p)^2 \quad \text{(by (3.1))}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\lambda \mathbf{ber}((Q^*|R|^2Q)^{2p} + (T|S^*|^2T^*)^{2p}) \\
&\quad + \frac{1}{2}(1-\lambda)\mathbf{ber}((Q^*|R|^2Q)^{2p} + (T|S^*|^2T^*)^{2p}) \quad (\text{by Lemma 3.1(1)}) \\
&= \frac{1}{2}\mathbf{ber}((Q^*|R|^2Q)^{2p} + (T|S^*|^2T^*)^{2p}),
\end{aligned}$$

which proves the second inequality. $\square$

Remark 3.5 Assume $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$ and $p \geq 1$. Applying inequality (3.5) we have

$$\begin{aligned}
\mathbf{ber}^{2p}(TSRQ) &\leq \min_{\lambda \in [0,1]} \left(\frac{1}{2}\lambda(\mathbf{ber}((Q^*|R|^2Q)^{2p} + (T|S^*|^2T^*)^{2p})) \right. \\
&\quad \left. + \frac{1}{2}(1-\lambda)\mathbf{ber}^p(TSRQ)\mathbf{ber}((Q^*|R|^2Q)^p + (T|S^*|^2T^*)^p) \right) \\
&\leq \max_{\lambda \in [0,1]} \left(\frac{1}{2}\lambda(\mathbf{ber}((Q^*|R|^2Q)^{2p} + (T|S^*|^2T^*)^{2p})) \right. \\
&\quad \left. + \frac{1}{2}(1-\lambda)\mathbf{ber}^p(TSRQ)\mathbf{ber}((Q^*|R|^2Q)^p + (T|S^*|^2T^*)^p) \right) \\
&\leq \frac{1}{2}\mathbf{ber}((Q^*|R|^2Q)^{2p} + (T|S^*|^2T^*)^{2p}).
\end{aligned}$$

Corollary 3.6 Suppose $M \in \mathcal{L}(\mathcal{H}(\Theta))$ and $s, t \geq 0$ with $s+t \geq 1$. Then

$$\begin{aligned}
\mathbf{ber}^{2p}(M|M|^{s+t-1}) &\leq \frac{1}{2}\lambda \mathbf{ber}(|M|^{4ps} + |M^*|^{4pt}) \\
&\quad + \frac{1}{2}(1-\lambda)\mathbf{ber}^p(M|M|^{s+t-1})\mathbf{ber}(|M|^{2\alpha s} + |M^*|^{2pt}) \\
&\leq \frac{1}{2}\mathbf{ber}(|M|^{4ps} + |M^*|^{4pt})
\end{aligned}$$

for all $p \geq 1$ and $\lambda \in [0, 1]$.

Proof Let $T = U$, $R = 1_{\mathcal{H}}$, $S = |M|^t$ and $Q = |M|^s$ such that $s+t \geq 1$ in inequality (3.5) where M and U are in $M = U|M|$, the polar decomposition of M . Then we have

$$TSRQ = U|M|^t|M|^s = U|M||M|^{s+t-1} = M|M|^{s+t-1},$$

also, we have $Q^*|R|^2Q = |M|^{2s}$ and $T|S^*|^2T^* = U|M|^{2t}U^* = |M^*|^{2t}$. $\square$

Applying Theorem 3.4 we get the following a refined Berezin-Schwarz inequality.

Corollary 3.7 Let $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$, $\lambda \in [0, 1]$ and $p \geq 1$. Then

$$\begin{aligned}
\mathbf{ber}^{2p}(TSRQ) &\leq \lambda \mathbf{ber}((Q^*|R|^2Q)^{2p})\mathbf{ber}((T|S^*|^2T^*)^{2p}) \\
&\quad + (1-\lambda)\mathbf{ber}^p(TSRQ)\mathbf{ber}((Q^*|R|^2Q)^p)\mathbf{ber}((T|S^*|^2T^*)^p) \\
&\leq \mathbf{ber}((Q^*|R|^2Q)^{2p})\mathbf{ber}((T|S^*|^2T^*)^{2p}).
\end{aligned} \tag{3.5}$$

Proof Assume $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$, $\lambda \in [0, 1]$ and $p \geq 1$. Employing inequality (3.5) we have

$$\begin{aligned} \mathbf{ber}^{2p}(TSRQ) &\leq \frac{1}{2}\lambda \mathbf{ber}((Q^*|R|^2Q)^{2p} + (T|S^*|^2T^*)^{2p}) \\ &\quad + \frac{1}{2}(1-\lambda)\mathbf{ber}^p(TSRQ)\mathbf{ber}((Q^*|R|^2Q)^p + (T|S^*|^2T^*)^p) \quad (\text{by (3.5)}) \\ &\leq \frac{1}{2}\lambda \left(\mathbf{ber}((Q^*|R|^2Q)^{2p}) + \mathbf{ber}((T|S^*|^2T^*)^{2p}) \right) \\ &\quad + \frac{1}{2}(1-\lambda)\mathbf{ber}^p(TSRQ) \left(\mathbf{ber}((Q^*|R|^2Q)^p) + \mathbf{ber}((T|S^*|^2T^*)^p) \right) \\ &\quad (\text{by the subadditive property of } \mathbf{ber}). \end{aligned} \quad (3.6)$$

Now, if we replace Q by $\sqrt{t}Q$ and T by $\frac{T}{\sqrt{t}}$ ($t > 0$) in (3.6) we get

$$\begin{aligned} \mathbf{ber}^{2p}(TSRQ) &\leq \frac{1}{2}\lambda \left(t^{2p}\mathbf{ber}((Q^*|R|^2Q)^{2p}) + \frac{1}{t^{2p}}\mathbf{ber}((T|S^*|^2T^*)^{2p}) \right) \\ &\quad + \frac{1}{2}(1-\lambda)\mathbf{ber}^p(TSRQ) \left(t^p\mathbf{ber}((Q^*|R|^2Q)^p) + \frac{1}{t^p}\mathbf{ber}((T|S^*|^2T^*)^p) \right). \end{aligned} \quad (3.7)$$

It follows from the recent inequality and

$$\begin{aligned} \min_{t>0} f(t) &= \min_{t>0} \left(t^{2p}\mathbf{ber}((Q^*|R|^2Q)^{2p}) + \frac{1}{t^{2p}}\mathbf{ber}((T|S^*|^2T^*)^{2p}) \right) \\ &= 2\mathbf{ber}((Q^*|R|^2Q)^{2p})\mathbf{ber}((T|S^*|^2T^*)^{2p}) \end{aligned}$$

and

$$\begin{aligned} \min_{t>0} g(t) &= \min_{t>0} \left(t^p\mathbf{ber}((Q^*|R|^2Q)^p) + \frac{1}{t^p}\mathbf{ber}((T|S^*|^2T^*)^p) \right) \\ &= 2\mathbf{ber}((Q^*|R|^2Q)^p)\mathbf{ber}((T|S^*|^2T^*)^p) \end{aligned}$$

that

$$\begin{aligned} \mathbf{ber}^{2p}(TSRQ) &\leq \lambda \mathbf{ber}((Q^*|R|^2Q)^{2p})\mathbf{ber}((T|S^*|^2T^*)^{2p}) \\ &\quad + (1-\lambda)\mathbf{ber}^p(TSRQ)\mathbf{ber}((Q^*|R|^2Q)^p)\mathbf{ber}((T|S^*|^2T^*)^p), \end{aligned} \quad (3.8)$$

where $\lambda \in [0, 1]$. Hence we get the first inequality. For the second inequality, using inequality (3.1) we have

$$\begin{aligned} \mathbf{ber}^p(TSRQ) &\leq \frac{1}{2}\mathbf{ber}((Q^*|R|^2Q)^p + (T|S^*|^2T^*)^p) \\ &\leq \frac{1}{2} \left(\mathbf{ber}((Q^*|R|^2Q)^p) + \mathbf{ber}((T|S^*|^2T^*)^p) \right), \end{aligned} \quad (3.9)$$

where $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$ and $p \geq 1$. If we replace Q by $\sqrt{t}Q$ and T by $\frac{T}{\sqrt{t}}$ ($t > 0$) in (3.9) we reach

$$\mathbf{ber}^p(TSRQ) \leq \frac{1}{2} \left(t^p\mathbf{ber}((Q^*|R|^2Q)^p) + \frac{1}{t^p}\mathbf{ber}((T|S^*|^2T^*)^p) \right). \quad (3.10)$$

It follows from $\min_{t>0} g(t) = 2\mathbf{ber}((Q^*|R|^2Q)^p)\mathbf{ber}((T|S^*|^2T^*)^p)$ that

$$\mathbf{ber}^p(TSRQ) \leq \mathbf{ber}((Q^*|R|^2Q)^p)\mathbf{ber}((T|S^*|^2T^*)^p). \quad (3.11)$$

Hence,

$$\begin{aligned} & \lambda \mathbf{ber}((Q^*|R|^2Q)^{2p})\mathbf{ber}((T|S^*|^2T^*)^{2p}) \\ & + (1-\lambda)\mathbf{ber}^p(TSRQ)\mathbf{ber}((Q^*|R|^2Q)^p)\mathbf{ber}((T|S^*|^2T^*)^p) \\ & \leq \lambda \mathbf{ber}((Q^*|R|^2Q)^{2p})\mathbf{ber}((T|S^*|^2T^*)^{2p}) \\ & + (1-\lambda)\mathbf{ber}^2((Q^*|R|^2Q)^p)\mathbf{ber}^2((T|S^*|^2T^*)^p) \quad (\text{by (3.11)}) \\ & \leq \lambda \mathbf{ber}((Q^*|R|^2Q)^{2p})\mathbf{ber}((T|S^*|^2T^*)^{2p}) \\ & + (1-\lambda)\mathbf{ber}((Q^*|R|^2Q)^{2p})\mathbf{ber}((T|S^*|^2T^*)^{2p}) \quad (\text{by Lemma 3.1(2)}) \\ & = \mathbf{ber}((Q^*|R|^2Q)^{2p})\mathbf{ber}((T|S^*|^2T^*)^{2p}), \end{aligned}$$

whence we get the second inequality. $\square$

With the same argument in the proof of Corollary 3.6 we have the following result.

Corollary 3.8 *Let $M \in \mathcal{L}(\mathcal{H}(\Theta))$, $\lambda \in [0, 1]$ and $p \geq 1$. Then*

$$\begin{aligned} \mathbf{ber}^{2p}(M|M|^{s+t-1}) & \leq \lambda \mathbf{ber}(|M|^{4ps})\mathbf{ber}(|M^*|^{4pt}) \\ & + (1-\lambda)\mathbf{ber}^p(M|M|^{s+t-1})\mathbf{ber}(|M|^{2ps})\mathbf{ber}(|M^*|^{2pt}) \\ & \leq \mathbf{ber}(|M|^{4ps})\mathbf{ber}(|M^*|^{4pt}), \end{aligned}$$

where $s, t \geq 0$ with $s+t \geq 1$.

In the following theorem, we show a reverse of the Schwarz-Berezin inequality.

Theorem 3.9 *Let $Q, S, R, T \in \mathcal{L}(\mathcal{H}(\Theta))$. Then*

$$\begin{aligned} & \mathbf{ber}(Q^*|R|^2Q)\mathbf{ber}(T|S^*|^2T^*) - \mathbf{ber}(TSRQ) \\ & \leq \mathbf{ber}(T|S^*|^2T^*) \min_{\alpha \in \mathbb{C}} \left(\mathbf{ber}(Q^*|R|^2Q + |\alpha|^2 T|S^*|^2T^* - \Re(\alpha TSRQ)) \right). \end{aligned}$$

Proof Let $Q, S, R, T \in \mathcal{L}(\mathcal{H}(\Theta))$ and $\alpha \in \mathbb{C}$. If we take $u = RQ\hat{k}_\nu$ and $v = S^*T^*\hat{k}_\nu$ in Lemma 1.1(1), then we have

$$\begin{aligned} |\langle u, v \rangle|^2 & = \langle RQ\hat{k}_\nu, S^*T^*\hat{k}_\nu \rangle = \langle TSRQ\hat{k}_\nu, \hat{k}_\nu \rangle \\ \|u\|^2 & = \langle RQ\hat{k}_\nu, RQ\hat{k}_\nu \rangle = \langle Q^*R^*RQ\hat{k}_\nu, \hat{k}_\nu \rangle = \langle Q^*|R|^2Q\hat{k}_\nu, \hat{k}_\nu \rangle, \\ \|v\|^2 & = \langle S^*T^*\hat{k}_\nu, S^*T^*\hat{k}_\nu \rangle = \langle TSS^*T^*\hat{k}_\nu, \hat{k}_\nu \rangle = \langle T|S^*|^2T^*\hat{k}_\nu, \hat{k}_\nu \rangle, \end{aligned}$$

and

$$\begin{aligned}
 & \|u - \alpha v\|^2 \\
 &= \langle RQ\widehat{k}_\nu - \alpha Q^*R^*\widehat{k}_\nu, RQ\widehat{k}_\nu - \alpha Q^*R^*\widehat{k}_\nu \rangle \\
 &= \langle RQ\widehat{k}_\nu, RQ\widehat{k}_\nu \rangle - \alpha \langle S^*T^*\widehat{k}_\nu, RQ\widehat{k}_\nu \rangle - \bar{\alpha} \langle RQ\widehat{k}_\nu, S^*T^*\widehat{k}_\nu \rangle + \alpha \bar{\alpha} \langle S^*T^*\widehat{k}_\nu, S^*T^*\widehat{k}_\nu \rangle \\
 &= \langle Q^*|R|^2 Q\widehat{k}_\nu, \widehat{k}_\nu \rangle + \alpha \bar{\alpha} \langle T|S^*|^2 T^*\widehat{k}_\nu, \widehat{k}_\nu \rangle - \alpha \langle Q^*R^*S^*T^*\widehat{k}_\nu, \widehat{k}_\nu \rangle - \bar{\alpha} \langle T S R Q\widehat{k}_\nu, \widehat{k}_\nu \rangle \\
 &= \left\langle (Q^*|R|^2 Q + \alpha \bar{\alpha} T|S^*|^2 T^* - \alpha Q^*R^*S^*T^* - \bar{\alpha} T S R Q)\widehat{k}_\nu, \widehat{k}_\nu \right\rangle \\
 &= \left\langle [Q^*|R|^2 Q + \alpha \bar{\alpha} T|S^*|^2 T^* - \Re(\alpha T S R Q)]\widehat{k}_\nu, \widehat{k}_\nu \right\rangle,
 \end{aligned}$$

where $\widehat{k}_\nu \in \mathcal{H}(\Theta)$ and $\alpha \in \mathbb{C}$. Utilizing Lemma 1.1(1) and these equations we deduce

$$\begin{aligned}
 & \langle Q^*|R|^2 Q\widehat{k}_\nu, \widehat{k}_\nu \rangle \langle T|S^*|^2 T^*\widehat{k}_\nu, \widehat{k}_\nu \rangle - \langle T S R Q\widehat{k}_\nu, \widehat{k}_\nu \rangle \\
 & \leq \langle T|S^*|^2 T^*\widehat{k}_\nu, \widehat{k}_\nu \rangle \left(\left\langle [Q^*|R|^2 Q + \alpha \bar{\alpha} T|S^*|^2 T^* - \Re(\alpha T S R Q)]\widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right),
 \end{aligned}$$

where $\widehat{k}_\nu \in \mathcal{H}(\Theta)$. If we take the supremum over all unit vectors $\widehat{k}_\nu \in \mathcal{H}(\Theta)$ we get

$$\begin{aligned}
 & \mathbf{ber}(Q^*|R|^2 Q) \mathbf{ber}(T|S^*|^2 T^*) - \mathbf{ber}(T S R Q) \\
 & \leq \mathbf{ber}(T|S^*|^2 T^*) \left(\mathbf{ber}(Q^*|R|^2 Q + \alpha \bar{\alpha} T|S^*|^2 T^* - \Re(\alpha T S R Q)) \right),
 \end{aligned}$$

where $\alpha \in \mathbb{C}$. It follows from α is arbitrary that we get the infimum on this inequality and we reach the desired result. $\square$

Using Lemma 1.1(2) and a same argument in the proof of Theorem 3.9 we have another reverse of the Schwarz-Berezin inequality.

Theorem 3.10 *Let $Q, S, R, T \in \mathcal{L}(\mathcal{H}(\Theta))$. Then*

$$\begin{aligned}
 & \mathbf{ber}(Q^*|R|^2 Q) \mathbf{ber}(T|S^*|^2 T^*) - \mathbf{ber}(T S R Q) \\
 & \leq \frac{1}{4} \mathbf{ber}(Q^*|R|^2 Q + T|S^*|^2 T^* - \Re(T S R Q)) \\
 & \quad \times \mathbf{ber}(Q^*|R|^2 Q + T|S^*|^2 T^* + \Re(T S R Q)).
 \end{aligned}$$

Lemma 3.11 *Let $Q_i, R_i, S_i, T_i \in \mathcal{L}(\mathcal{H}(\Theta))$ ($i = 1, \dots, n$). Then*

$$\begin{aligned}
 & \left| \left\langle \sum_{i=1}^n T_i S_i R_i Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right| \leq 2^{-\frac{1}{p}} \sum_{i=1}^n \left(\left\langle (Q_i^*|R_i|^2 Q_i)^p + (T_i|S_i^*|^2 T_i^*)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right)^{\frac{1}{p}} \\
 & \quad - \frac{1}{2} \sum_{i=1}^n \left(\left\langle Q_i^*|R_i|^2 Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} - \left\langle T_i|S_i^*|^2 T_i^* \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} \right)^2
 \end{aligned} \tag{3.12}$$

Proof Let $\widehat{k}_\nu \in \mathcal{H}(\Theta)$. In [20], the authors showed the numerical inequality

$$(u^{\frac{1}{2}} v^{\frac{1}{2}}) + \frac{1}{2} (u^{\frac{1}{2}} - v^{\frac{1}{2}})^2 \leq 2^{-\frac{1}{p}} (u^p + v^p)^{\frac{1}{p}} \quad \text{for } u, v > 0 \quad \text{and } p \geq 1. \tag{3.13}$$

Using this inequality we have

$$\begin{aligned}
 \left| \left\langle \sum_{i=1}^n T_i S_i R_i Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right| &\leq \sum_{i=1}^n \left| \left\langle T_i S_i R_i Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right| \\
 &\leq \sum_{i=1}^n \sqrt{\left\langle Q_i^* |R_i|^2 Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \left\langle T_i |S_i^*|^2 T_i^* \widehat{k}_\nu, \widehat{k}_\nu \right\rangle} \quad (\text{by (1.5)}) \\
 &\leq 2^{-\frac{1}{p}} \sum_{i=1}^n \left(\left\langle Q_i^* |R_i|^2 Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^p + \left\langle T_i |S_i^*|^2 T_i^* \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^p \right)^{\frac{1}{p}} \\
 &\quad - \frac{1}{2} \sum_{i=1}^n \left(\left\langle Q_i^* |R_i|^2 Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} - \left\langle T_i |S_i^*|^2 T_i^* \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} \right)^2 \\
 &\quad (\text{by (3.13)}) \\
 &\leq 2^{-\frac{1}{p}} \sum_{i=1}^n \left(\left\langle (Q_i^* |R_i|^2 Q_i)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle + \left\langle (T_i |S_i^*|^2 T_i^*)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right)^{\frac{1}{p}} \\
 &\quad - \frac{1}{2} \sum_{i=1}^n \left(\left\langle Q_i^* |R_i|^2 Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} - \left\langle T_i |S_i^*|^2 T_i^* \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} \right)^2 \\
 &\quad (\text{by (1.1)}) \\
 &\leq 2^{-\frac{1}{p}} \sum_{i=1}^n \left(\left\langle (Q_i^* |R_i|^2 Q_i)^p + (T_i |S_i^*|^2 T_i^*)^p \widehat{k}_\nu, \widehat{k}_\nu \right\rangle \right)^{\frac{1}{p}} \\
 &\quad - \frac{1}{2} \sum_{i=1}^n \left(\left\langle Q_i^* |R_i|^2 Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} - \left\langle T_i |S_i^*|^2 T_i^* \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} \right)^2.
 \end{aligned}$$

□

Corollary 3.12 Let $Q_i, R_i, S_i, T_i \in \mathcal{L}(\mathcal{H}(\Theta))$ ($i = 1, \dots, n$). Then

$$\begin{aligned}
 &\text{ber} \left(\sum_{i=1}^n T_i S_i R_i Q_i \right) \\
 &\leq 2^{-\frac{1}{p}} \sum_{i=1}^n \text{ber} \left((Q_i^* |R_i|^2 Q_i)^p + (T_i |S_i^*|^2 T_i^*)^p \right) \\
 &\quad - \frac{1}{2} \sum_{i=1}^n \inf_{\widehat{k}_\nu \in \mathcal{H}(\Theta)} \left(\left\langle Q_i^* |R_i|^2 Q_i \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} - \left\langle T_i |S_i^*|^2 T_i^* \widehat{k}_\nu, \widehat{k}_\nu \right\rangle^{\frac{1}{2}} \right)^2.
 \end{aligned}$$

Proof If we taking the supremum over all unit vectors $\widehat{k}_\nu \in \mathcal{H}(\Theta)$ in inequality (3.12) we get desired result. □

Using Lemma 1.3 we obtain the next result which is a refinement and a generalization of inequality (3.1).

Theorem 3.13 Let $Q, R, S, T \in \mathcal{L}(\mathcal{H}(\Theta))$ and $g : I \subseteq \mathcal{R}^+ \rightarrow \mathcal{R}$ be an increasing convex function. Then

$$\begin{aligned} & g(\mathbf{ber}(TSRQ)) \\ & \leq \mathbf{ber} \left(\int_0^1 g(tQ^*|R|^2Q + (1-t)T|S^*|^2T^*) dt \right) \\ & \leq \frac{1}{2} \mathbf{ber} \left(g((1-\lambda)Q^*|R|^2Q + \lambda T|S^*|^2T^*) + (1-\lambda)g(T|S^*|^2T^*) + \lambda g(Q^*|R|^2Q) \right) \\ & \leq \frac{1}{2} \mathbf{ber} \left(g(Q^*|R|^2Q) + g(T|S^*|^2T^*) \right), \end{aligned}$$

where $\lambda \in [0, 1]$.

Proof Let $\hat{k}_\lambda \in \mathcal{H}(\Theta)$. We have

$$\begin{aligned} g \left(\left| \left\langle TSRQ\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| \right) & \leq g \left(\sqrt{\left\langle Q^*|R|^2Q\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \left\langle T|S^*|^2T^*\hat{k}_\lambda, \hat{k}_\lambda \right\rangle} \right) \quad (\text{by (1.5)}) \\ & \leq g \left(\frac{\left\langle Q^*|R|^2Q\hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \left\langle T|S^*|^2T^*\hat{k}_\lambda, \hat{k}_\lambda \right\rangle}{2} \right) \\ & \quad (\text{by the AM-GM inequality}) \\ & \leq \left\langle \int_0^1 g((1-t)Q^*|R|^2Q + tT|S^*|^2T^*) dt \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \quad (\text{by (1.6)}) \\ & \leq \frac{1}{2} \left\langle [g((1-\lambda)Q^*|R|^2Q + \lambda T|S^*|^2T^*) \right. \\ & \quad \left. + (1-\lambda)g(T|S^*|^2T^*) + \lambda g(Q^*|R|^2Q)] \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \quad (\text{by (1.7)}) \\ & \leq \left\langle \left(\frac{g(Q^*|R|^2Q) + g(T|S^*|^2T^*)}{2} \right) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \quad (\text{by (1.8)}). \end{aligned}$$

Taking the supremum over all unit vectors $\hat{k}_\nu \in \mathcal{H}(\Theta)$ we get desired result. $\square$

It follows from $g(x) = x^p$ ($p \geq 1$) is increasing convex that we have the next corollary.

Corollary 3.14 Let $T, Q, R, S \in \mathcal{L}(\mathcal{H}(\Theta))$. Then

$$\begin{aligned} & \mathbf{ber}^p(TSRQ) \\ & \leq \mathbf{ber} \left(\int_0^1 (tQ^*|R|^2Q + (1-t)T|S^*|^2T^*)^p dt \right) \\ & \leq \frac{1}{2} \mathbf{ber} \left(((1-\lambda)Q^*|R|^2Q + \lambda T|S^*|^2T^*)^p + (1-\lambda)(T|S^*|^2T^*)^p + \lambda(Q^*|R|^2Q)^p \right) \\ & \leq \frac{1}{2} \mathbf{ber} \left((Q^*|R|^2Q)^p + (T|S^*|^2T^*)^p \right), \end{aligned}$$

in which $p \geq 1$ and $\lambda \in [0, 1]$.

Acknowledgment

The author would like to sincerely thank the anonymous referees for some useful comments and suggestions. The first author also would like to thank the Tusi Mathematical Research Group (TMRG).

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