

RESEARCH

Open Access



A new mean-Berezin norm for operators in reproducing kernel Hilbert spaces

Mojtaba Bakherad^{1*}

*Correspondence:
mojtaba.bakherad@yahoo.com

¹Department of Mathematics,
University of Sistan and
Baluchestan, Zahedan, Iran

Abstract

A functional Hilbert space is defined as the Hilbert space $\mathcal{K}$ of complex-valued functions defined on a set Θ . In this space, the evaluation functionals $\psi_\varepsilon(h) = h(\varepsilon)$, for $\varepsilon \in \Theta$, are continuous on $\mathcal{K}$. In this paper, we introduce the mean-Berezin norm $\|\cdot\|_{\rho_t}$, defined as follows:

$$\|G\|_{M_{\rho_t}} = \sup_{\kappa, \xi \in \Theta} \left\{ \left(|\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle| \right)^{\rho_t} \left(|\langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle| \right)^{\rho_t} \right\}, \quad \text{for } 0 \leq t \leq 1,$$

where $G \in \mathcal{O}(\mathcal{K})$ and ρ_t represents an interpolation path of a symmetric mean ρ . Using the definition of the mean-Berezin norm, we explore some related new inequalities. For instance, if $G \in \mathcal{O}(\mathcal{K})$, then

$$\frac{1}{2} \sqrt{\| |G|^{2r} + |G^*|^{2(1-r)} \|_{\text{ber}}} \geq \|G\|_{M_\rho},$$

where $0 \leq r \leq 1$ and ρ is a mean such that $\rho \leq \nabla$.

Mathematics Subject Classification: 47B32; 47B34

Keywords: Means; Berezin set; Berezin symbol; Berezin number; Berezin norm; Interpolation path

1 Introduction and preliminaries

A binary function ρ defined on $[0, +\infty)$ is definitively classified as a mean if it satisfies the following conditions:

1. For any $\lambda > 0$, the inequality $(\lambda x) \rho (\lambda y) \geq \lambda (x \rho y)$ holds.
2. The function ρ is continuous in both variables.
3. If $x \geq z$ and $y \geq w$, then $x \rho y \geq z \rho w$.
4. If $x \geq y$, then $x \geq (x \rho y) \geq y$.

These conditions define the properties that a mean must have. For example, if $s \in [0, 1]$, the weighted geometric mean is defined as $x \sharp_s y = x^{1-s} y^s$. When $s = \frac{1}{2}$, this reduces to the geometric mean, represented as $x \sharp y$. We say a mean ρ is symmetric if it satisfies the condition $x \rho y = y \rho x$ for all positive numbers x and y . If ρ is a symmetric mean, then a parameter-

© The Author(s) 2025. **Open Access** This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License, which permits any non-commercial use, sharing, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if you modified the licensed material. You do not have permission under this licence to share adapted material derived from this article or parts of it. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by-nc-nd/4.0/>.

ized mean ρ_t , where $0 \leq t \leq 1$, is referred to as an interpolational path for ρ if it adheres to the following criteria:

1. $x\rho_0 y = x$, $x\rho_{1/2} y = x\rho y$, and $x\rho_1 y = y$;
2. $(x\rho_p y)\rho(x\rho_q y) = x\rho_{\frac{p+q}{2}} y$ for all $p, q \in [0, 1]$;
3. The mapping $t \in [0, 1] \mapsto x\rho_t y$ is continuous for each fixed x and y ;
4. Each component of the mean ρ_t is increasing for $t \in [0, 1]$.

For example, the power means are defined as follows:

$$xp_r y = \left(\frac{x^r + y^r}{2} \right)^{\frac{1}{r}} \quad (r \in [-1, 1])$$

The interpolational paths associated with these means can be expressed as:

$$xp_{r,t} y = ((1-t)x^r + ty^r)^{\frac{1}{r}} \quad (t \in [0, 1]).$$

In particular, $xp_{1,\lambda} y = x\nabla_\lambda y = (1-\lambda)x + \lambda y$ is the weighted arithmetic mean, $ap_{0,\lambda} b = a\sharp_\lambda b = a^{1-\lambda}b^\lambda$ is the weighted geometric mean and $xp_{-1,\lambda} y = x!_\lambda y = ((1-\lambda)x^{-1} + \lambda y^{-1})^{-1}$ is the weighted harmonic mean. It is well-known the inequality $x!_\lambda y \leq x\sharp_\lambda y \leq x\nabla_\lambda y$, where x and y are positive numbers and $\lambda \in [0, 1]$; see [30] and the references therein.

Let $\mathcal{O}(\mathcal{K})$ denote the C^* -algebra comprising all bounded linear operators defined on a complex Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle)$. Within $\mathcal{O}(\mathcal{K})$, the identity operator is represented as $I_{\mathcal{K}}$.

A functional Hilbert space is defined as the Hilbert space of complex-valued functions on a set $\Theta \subseteq \mathbb{C}$, in which the evaluation functionals $\psi_\varepsilon(h) = h(\varepsilon)$, for $\varepsilon \in \Theta$, are continuous on the space $\mathcal{K}$. In accordance with the Riesz representation theorem, there exists a unique element $k_\kappa \in \mathcal{K}$ such that for every function $f \in \mathcal{K}$ and for all $\kappa \in \Theta$, the following holds: $f(\kappa) = \langle f, k_\kappa \rangle$.

The function k , defined on the Cartesian product $\Theta \times \Theta$ by $k(w, \kappa) = k_\kappa(w)$, is referred to as the reproducing kernel of the space $\mathcal{K}$ (see [1, 4, 6, 16, 29] and associated references). It has been demonstrated that $k_\kappa(w)$ can be represented in the following manner:

$$k_\kappa(w) = \sum_{n=1}^{\infty} \overline{l_n(\kappa)} l_n(w)$$

for any orthonormal basis $\{l_n\}_{n \geq 1}$ of $\mathcal{K}$ (refer to [34]), it is established that the reproducing kernel of $\mathcal{K}^2$ is represented by the following function

$$k_\kappa(w) = \sum_{n=1}^{\infty} \overline{\kappa_n} w^n = (1 - \overline{\kappa} w)^{-1}, \quad \kappa \in \mathbb{Q} = \{w \in \mathbb{C} : |w| < 1\}.$$

Let $\hat{k}_\kappa = \frac{k_\kappa}{\|k_\kappa\|}$ represent the normalized reproducing kernel associated with the space $\mathcal{K}$. For a specified operator $G \in \mathcal{O}(\mathcal{K})$, the Berezin symbol of G is defined as the bounded function $\tilde{G}$ on Θ given by the expression

$$\tilde{G}(\kappa) = \langle G\hat{k}_\kappa(w), \hat{k}_\kappa(w) \rangle, \quad \kappa \in \Theta.$$

A significant property of the Berezin symbol is that for any operators $G, H \in \mathcal{O}(\mathcal{K})$, if $\tilde{G}(\kappa) = \tilde{H}(\kappa)$ holds for all $\kappa \in \Theta$, then it follows that $G = H$ (this holds particularly when

$\mathcal{K}$ consists of analytic functions; see Zhu [35]). For additional information, please refer to the works cited in [3, 7–10, 12–14, 17, 19–33].

The mapping from G to $\tilde{G}$ is injective [15]. Furthermore, the Berezin set and the Berezin number (often referred to as the “radius”) of an operator G are respectively defined as follows:

$$\mathbf{Ber}(G) = \{\tilde{G}(\kappa) : \kappa \in \Theta\}$$

and

$$\mathbf{ber}(G) = \sup_{\kappa \in \Theta} |\tilde{G}(\kappa)|.$$

The Berezin norm of an operator $G \in \mathcal{O}(\mathcal{K})$ is defined by

$$\|G\|_{\mathbf{ber}} = \sup_{\kappa, \xi \in \Theta} |\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle|.$$

For an operator $G \in \mathcal{O}(\mathcal{K})$, the Crawford Berezin numbers are defined as follows [18, 25]:

$$\tilde{\alpha}(G) = \inf_{\kappa \in \Theta} |\tilde{G}(\kappa)| \quad \text{and} \quad \tilde{\beta}(G) = \inf_{\kappa, \xi \in \Theta} |\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle|.$$

In a recent paper [28], the authors defined the t -Berezin norm on $\mathcal{O}(\mathcal{K})$ as follows:

$$\|G\|_{M_t\text{-ber}} = \sup_{\kappa, \xi \in \Theta} \left\{ t |\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle| + (1-t) |\langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle| \right\}.$$

The t -Berezin norm constitutes a valid norm on $\mathcal{O}(\mathcal{K})$. This indicates that the t -Berezin norm is applicable to well-established functionals associated with Hilbert spaces, such as Hardy and Bergman spaces. Furthermore, the t -Berezin norm adheres to a series of inequalities, which enhances its utility and relevance in the context of functional analysis as follows

$$\mathbf{ber}(G) \leq \|G\|_{M_t\text{-ber}} \leq \|G\|_{\mathbf{ber}} \quad \text{for } t \in [0, 1].$$

We define a new norm based on means as follows.

Definition 1 Let G belong to $\mathcal{O}(\mathcal{K})$, and let ρ_t represent an interpolational path of a symmetric mean ρ . We confidently define the mean-Berezin norm as follows:

$$\|G\|_{M_{\rho_t}} = \sup_{\kappa, \xi \in \Theta} \left\{ \left(|\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \rho_t \left(|\langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \right\} \quad \text{for } 0 \leq t \leq 1.$$

It is evident that for the special case where $\rho_t = \nabla_t$ for $0 \leq t \leq 1$, we have $\|\cdot\|_{M_{\rho(1-t)}} = \|\cdot\|_{M_t\text{-ber}}$. Furthermore, it follows from the relationship $|\langle G\hat{k}_\kappa, \hat{k}_\kappa \rangle| = |\langle G\hat{k}_\kappa, \hat{k}_\kappa \rangle| \rho_t |\langle G^*\hat{k}_\kappa, \hat{k}_\kappa \rangle|$ that we have:

$$\begin{aligned} \mathbf{ber}(G) &= \sup_{\kappa \in \Theta} |\langle G\hat{k}_\kappa, \hat{k}_\kappa \rangle| \\ &= \sup_{\kappa \in \Theta} \left\{ |\langle G\hat{k}_\kappa, \hat{k}_\kappa \rangle| \rho_t |\langle G^*\hat{k}_\kappa, \hat{k}_\kappa \rangle| \right\} \end{aligned}$$

$$\leq \sup_{\kappa, \xi \in \Theta} \left\{ \left| \langle G\hat{k}_\kappa, \hat{k}_\xi \rangle \right| \rho_t \left| \langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle \right| \right\} = \|G\|_{M_{\rho_t}} \quad \text{for all } 0 \leq t \leq 1 \quad (1)$$

and

$$\begin{aligned} \|G\|_{M_{\rho_t}} &= \sup_{\kappa, \xi \in \Theta} \left\{ \left| \langle G\hat{k}_\kappa, \hat{k}_\xi \rangle \right| \rho_t \left| \langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle \right| \right\} \\ &\leq (\|G\|_{\text{ber}}) \rho_t (\|G\|_{\text{ber}}) \\ &= \|G\|_{\text{ber}} \quad \text{for all } 0 \leq t \leq 1. \end{aligned}$$

Therefore, $\text{ber}(G) \leq \|G\|_{M_{\rho_t}} \leq \|G\|_{\text{ber}}$ for all $G \in \mathcal{O}(\mathcal{K})$ and all $0 \leq t \leq 1$.

Example 2 In the context of the Hardy-Hilbert space, denoted as $\mathcal{H}^2(\mathbb{Q})$, which is established over the unit disk $\mathbb{Q} = \{w \in \mathbb{C} : |w| < 1\}$, the following definition applies:

$$\mathcal{H}^2(\mathbb{Q}) = \{g : \mathbb{Q} \rightarrow \mathbb{C} : g(w) = \sum_{n=0}^{\infty} \xi_n w^n \text{ and } \sum_{n=0}^{\infty} |\xi_n|^2 < \infty\}.$$

It is possible to associate $\mathcal{H}^2(\mathbb{Q})$ with the sequence space $\ell^2(\mathbb{N})$ due to the correspondence established by:

$$g(w) = \sum_{n=0}^{\infty} \xi_n w^n \leftrightarrow (\xi_0, \xi_1, \xi_2, \dots).$$

The reproducing kernel of the Hardy space $\mathcal{H}^2(\mathbb{Q})$ is defined by the function $k_\kappa(w) = \sum_{n=0}^{\infty} \bar{\kappa}^n w^n = (1 - \bar{\kappa}w)^{-1}$, where κ is an element of $\mathbb{Q}$. The norm of the kernel is computed as follows:

$$\|k_\kappa\|^2 = \langle k_\kappa, k_\kappa \rangle = k_\kappa(\kappa) = (1 - \bar{\kappa}\kappa)^{-1} = (1 - |\kappa|^2)^{-1},$$

for any κ belonging to the unit disk $\mathbb{Q}$. Additionally, on $\ell^2(\mathbb{N})$, we consider the unilateral right shift operator. Thus, for any $\kappa \in \mathbb{Q}$, we have

$$\begin{aligned} \left| \langle U\hat{k}_\kappa(z), \hat{k}_\xi(z) \rangle \right| &= \frac{1}{\|\hat{k}_\kappa\| \|\hat{k}_\xi\|} \left| \langle Uk_\kappa(z), k_\xi(z) \rangle \right| \\ &= \frac{1}{\|\hat{k}_\kappa\| \|\hat{k}_\xi\|} \left| \left\langle U \left(\sum_{n=0}^{\infty} \bar{\kappa}^n z^n \right), \left(\sum_{n=0}^{\infty} \bar{\xi}^n z^n \right) \right\rangle \right| \\ &= \frac{1}{\|\hat{k}_\kappa\| \|\hat{k}_\xi\|} \left| \left\langle U(1, \bar{\kappa}, \bar{\kappa}^2, \dots), (1, \bar{\xi}, \bar{\xi}^2, \dots) \right\rangle \right| \\ &= \frac{1}{\|\hat{k}_\kappa\| \|\hat{k}_\xi\|} \left| \left\langle (0, 1, \bar{\kappa}, \bar{\kappa}^2, \dots), (1, \bar{\xi}, \bar{\xi}^2, \dots) \right\rangle \right| \\ &= \frac{1}{\|\hat{k}_\kappa\| \|\hat{k}_\xi\|} \left| \sum_{j=0}^{\infty} \bar{\kappa}^j \bar{\xi}^{j+1} \right| = \frac{1}{\|\hat{k}_\kappa\| \|\hat{k}_\xi\|} \left(\frac{1}{|1 - \bar{\kappa}\xi|} |\xi| \right), \end{aligned}$$

and similarly $\left| \left\langle U^* \hat{k}_\kappa(z), \hat{k}_\xi(z) \right\rangle \right| = \frac{1}{\|\hat{k}_\kappa\| \|\hat{k}_\xi\|} \left(\frac{1}{|1 - \bar{\kappa}\xi|} |\kappa| \right)$; see [5]. Moreover, it follows from U is an isometry, that we have $\|U \hat{k}_\kappa\|^2 = \|\hat{k}_\kappa\|^2 = 1$. Then, by the monotonicity of ρ_t , we obtain that

$$\begin{aligned} 1 &= \sup_{\kappa \in \mathbb{Q}} |\kappa| = \sup_{\kappa \in \mathbb{Q}} |\tilde{U}(\kappa)| \\ &\leq \|U\|_{M_{\rho_t}} \\ &= \sup_{\kappa, \xi \in \Theta} \left\{ \left| \langle U \hat{k}_\kappa, \hat{k}_\xi \rangle \right| \rho_t \left| \langle U^* \hat{k}_\kappa, \hat{k}_\xi \rangle \right| \right\} \\ &= \sup_{\kappa, \xi \in \Theta} \left\{ \left(\frac{|\xi|}{\|\hat{k}_\kappa\| \|\hat{k}_\xi\| |1 - \bar{\kappa}\xi|} \right) \rho_t \left(\frac{|\kappa|}{\|\hat{k}_\kappa\| \|\hat{k}_\xi\| |1 - \bar{\kappa}\xi|} \right) \right\} \\ &\leq \|U\|_{\text{ber}} \\ &\leq \sup_{\kappa \in \mathbb{Q}} \|U \hat{k}_\kappa\| = 1 \end{aligned}$$

for $t \in [0, 1]$. Hence $\|U\|_{M_{\rho_t}} = 1$ for any $t \in [0, 1]$.

Example 3 We consider the Hardy-Hilbert space, $\mathcal{H}^2(\mathbb{Q})$, defined over the unit disk $\mathbb{Q} = \{y \in \mathbb{C} : |y| < 1\}$. For every $y \in \mathbb{Q}$, we define the operator $G_y : \mathcal{H}^2(\mathbb{Q}) \rightarrow \mathcal{H}^2(\mathbb{Q})$ by $G_y(g) = yg$ for all $g \in \mathcal{H}^2(\mathbb{Q})$. Then, we have

$$\begin{aligned} 1 &= \text{ber}(G_y) \\ &\leq \|G_y\|_{M_{\rho_t}} \\ &= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ \left| \langle G_y \hat{k}_\kappa, \hat{k}_\xi \rangle \right| \rho_t \left| \langle G_y^* \hat{k}_\kappa, \hat{k}_\xi \rangle \right| \right\} \\ &= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ \left(\sqrt{1 - |\kappa|^2} \sqrt{1 - |\xi|^2} \frac{|\kappa|}{|1 - \bar{\kappa}\xi|} \right) \rho_t \left(\sqrt{1 - |\kappa|^2} \sqrt{1 - |\xi|^2} \frac{|\xi|}{|1 - \bar{\kappa}\xi|} \right) \right\} \\ &\leq \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ \left(\sqrt{1 - |\kappa|^2} \sqrt{1 - |\xi|^2} \frac{|\kappa|}{|1 - \bar{\kappa}\xi|} \right) \right\} \\ &= \|G_y\|_{\text{ber}}. \end{aligned}$$

In this paper, we present a novel quantity derived from the definition outlined above and establish significant results related to it. Furthermore, we pinpoint several inequalities associated with the Berezin number and the Berezin norm. The main concepts explored in this paper are firmly grounded in the influential works of [5, 11], and [28].

2 Main results

The following well-established lemmas are vital for substantiating our results.

Lemma 4 [26, 27] *If $G \in \mathcal{O}(\mathcal{K})$ is a self-adjoint operator such that its spectrum is included in an interval J and $\kappa \in \Theta$, then for a convex function $h : J \rightarrow \mathbb{R}$, we have*

$$h(\langle G \hat{k}_\kappa, \hat{k}_\kappa \rangle) \leq \langle h(G) \hat{k}_\kappa, \hat{k}_\kappa \rangle.$$

Lemma 5 [24] *If $G \in \mathcal{O}(\mathcal{K})$ and $k, l: [0, \infty) \rightarrow \mathbb{R}$ are non-negative continuous functions such that $k(s)l(s) = s$ for $s \geq 0$, then*

$$|\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle|^2 \leq \langle k^2(|G|)\hat{k}_\kappa, \hat{k}_\kappa \rangle \langle l^2(|G^*|)\hat{k}_\xi, \hat{k}_\xi \rangle$$

for all $\kappa, \xi \in \Theta$.

We begin our study by thoroughly analyzing the fundamental properties of the mean-Berezin norm, a clear consequence of its definition.

Proposition 6 *Let $G \in \mathcal{O}(\mathcal{K})$ and ρ_t, κ_s be interpolational paths of symmetric means ρ and κ . Then*

- (1) $\|G\|_{M_{\rho_t}} = \|G^*\|_{M_{\rho_t}}$;
- (2) $\|G\|_{M_{\rho_0}} = \|G\|_{M_{\rho_1}} = \|G\|_{\text{ber}}$;
- (3) $\|\lambda G\|_{M_{\rho_t}} = |\lambda| \|G\|_{M_{\rho_t}}$ for all $\lambda \in \mathbb{C}$;
- (4) If $t \in [0, 1]$, then $\|G\|_{M_{\rho_t}} = 0$ if and only if $G = 0$;
- (5) If $\rho_t \leq \kappa_s$, then $\|G\|_{M_{\rho_t}} \leq \|G\|_{M_{\kappa_s}}$ for all $s, t \in [0, 1]$;
- (6) $\|G\|_{M_{\rho_t}} = \|G\|_{M_{\rho_{(1-t)}}}$ for all $t \in [0, 1]$.

In Proposition 6, we have established that the norm $\|\cdot\|_{M_{\rho_t}}$ (for $0 \leq t \leq 1$) fulfills the criteria of a semi-norm, with the exception of the triangle inequality, which may not hold. Notably, for the interpolational paths M_{ρ_t} ($0 \leq t \leq 1$), which have the following property:

$$(x + y)\rho_t(z + w) = (x\rho_t z) + (y\rho_t w) \quad \text{for all } x, y, z, w \geq 0, \quad (\star)$$

we can state the following proposition.

Proposition 7 *Let $G, H \in \mathcal{O}(\mathcal{K})$ and ρ_t ($t \in [0, 1]$) be an interpolational path of symmetric means ρ , which has the property $(\star)$. Then*

$$\|G + H\|_{M_{\rho_t}} \leq \|G\|_{M_{\rho_t}} + \|H\|_{M_{\rho_t}}.$$

Proof Let $\kappa, \xi \in \Theta$. Then, we have

$$\begin{aligned} & \left(|\langle (G + H)\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \rho_t \left(|\langle (G^* + H^*)\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \\ & \leq \left(|\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle| + |\langle H\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \rho_t \left(|\langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle| + |\langle H^*\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \\ & \quad \text{(by the property (3) of means)} \\ & = \left(|\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \rho_t \left(|\langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) + \left(|\langle H\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \rho_t \left(|\langle H^*\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \\ & \quad \text{(by the property } (\star)) \\ & \leq \|G\|_{M_{\rho_t}} + \|H\|_{M_{\rho_t}}. \end{aligned}$$

Therefore,

$$\begin{aligned}\|G + H\|_{M_{\rho_t}} &= \sup_{\kappa, \xi \in \Theta} \left\{ \left(|\langle (G + H)\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \rho_t \left(|\langle (G^* + H^*)\hat{k}_\kappa, \hat{k}_\xi \rangle| \right) \right\} \\ &\leq \|G\|_{M_{\rho_t}} + \|H\|_{M_{\rho_t}}.\end{aligned}\quad \square$$

Remark 8 Assume that $G, H \in \mathcal{O}(\mathcal{K})$. It is trivial, the weighted arithmetic means $\rho_t = \nabla_t$ ($0 \leq t \leq 1$) have the property (✕). Then, we have

$$\|G\|_{M_{\nabla_t}} + \|H\|_{M_{\nabla_t}} \geq \|G + H\|_{M_{\nabla_t}}.$$

We will now explore an equivalent characterization of the mean-Berezin norm and the Berezin norm.

Theorem 9 Let $G \in \mathcal{O}(\mathcal{K})$ and ρ_t ($t \in [0, 1]$) be an interpolational path of symmetric means ρ . Let

- (1) $\|G\|_{M_{\rho_t}} = \|G\|_{\text{ber}}$, and
- (2) There exists sequences $\{\kappa_n\}$ and $\{\xi_n\}$ in Θ such that

$$\lim_{n \rightarrow \infty} |\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| = \lim_{n \rightarrow \infty} |\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| = \|G\|_{\text{ber}}.$$

Then, the statements (1) and (2) are equivalent.

Proof Assuming that (1) is valid, according to the definition of the supremum, it follows that there exist sequences $\{\kappa_n\}$ and $\{\xi_n\}$ within the set Θ such that

$$\|G\|_{M_{\rho_t}} = \lim_{n \rightarrow \infty} |\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \rho_t |\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle|.$$

Hence,

$$\begin{aligned}\|G\|_{\text{ber}} &= \|G\|_{M_{\rho_t}} \\ &= \lim_{n \rightarrow \infty} \left(|\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \rho_t |\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \right) \\ &= \left(\lim_{n \rightarrow \infty} |\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \right) \rho_t \lim_{n \rightarrow \infty} \left(|\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \right) \\ &\leq (\|G\|_{\text{ber}}) \rho_t (\|G\|_{\text{ber}}) = \|G\|_{\text{ber}},\end{aligned}$$

whence,

$$\|G\|_{\text{ber}} = \left(\lim_{n \rightarrow \infty} |\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \right) \rho_t \lim_{n \rightarrow \infty} \left(|\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \right).$$

For $t = 0$, we have $\|G\|_{\text{ber}} = \lim_{n \rightarrow \infty} |\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle|$ and for $t = 1$, we have $\|G\|_{\text{ber}} = \lim_{n \rightarrow \infty} |\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle|$. Therefore,

$$\|G\|_{\text{ber}} = \lim_{n \rightarrow \infty} |\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| = \lim_{n \rightarrow \infty} |\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle|.$$

Now, let us consider the case where (2) holds. If

$$\|G\|_{\text{ber}} = \lim_{n \rightarrow \infty} |\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| = \lim_{n \rightarrow \infty} |\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle|,$$

then by the definition of the mean-Berezin norm, we have

$$\begin{aligned} \|G\|_{M_{\rho_t}} &\geq \lim_{n \rightarrow \infty} \left(|\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \rho_t |\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \right) \\ &= \left(\lim_{n \rightarrow \infty} |\langle G\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \right) \rho_t \left(\lim_{n \rightarrow \infty} |\langle G^*\hat{k}_{\kappa_n}, \hat{k}_{\xi_n} \rangle| \right) \\ &= (\|G\|_{\text{ber}}) \rho_t (\|G\|_{\text{ber}}) \\ &= \|G\|_{\text{ber}}. \end{aligned}$$

Therefore, $\|G\|_{M_{\rho_t}} = \|G\|_{\text{ber}}$. □

In the following theorem, we establish bounds for the norm $\|\cdot\|_{M_{\rho_t}}$.

Theorem 10 Let $G \in \mathcal{O}(\mathcal{K})$ and $0 \leq t \leq 1$. Then,

$$\begin{aligned} &\max\{(\|G\|_{\text{ber}})\rho_t(\tilde{C}(G)), (\tilde{C}(G))\rho_t(\|G\|_{\text{ber}})\} \\ &\leq \|G\|_{M_{\rho_t}} \\ &\leq \min \left\{ \|G^*G\|_{\text{ber}}^{\frac{1}{2}}, \|GG^*\|_{\text{ber}}^{\frac{1}{2}} \right\}. \end{aligned}$$

Proof Let $\kappa \in \Theta$ and $0 \leq t \leq 1$. Then, it follows that

$$|\langle G\hat{k}_{\kappa}, \hat{k}_{\xi} \rangle| \rho_t |\langle G^*\hat{k}_{\kappa}, \hat{k}_{\xi} \rangle| \geq (\tilde{C}(G))\rho_t |\langle G^*\hat{k}_{\kappa}, \hat{k}_{\xi} \rangle|.$$

Hence

$$\begin{aligned} \|G\|_{M_{\rho_t}} &= \sup_{\kappa, \xi \in \Theta} \left\{ |\langle G\hat{k}_{\kappa}, \hat{k}_{\xi} \rangle| \rho_t |\langle G^*\hat{k}_{\kappa}, \hat{k}_{\xi} \rangle| \right\} \\ &\geq (\tilde{C}(G))\rho_t |\langle G^*\hat{k}_{\kappa}, \hat{k}_{\xi} \rangle| \\ &= (\tilde{C}(G))\rho_t |\langle \hat{k}_{\kappa}, G\hat{k}_{\xi} \rangle|. \end{aligned}$$

Taking the supremum over all vectors $\kappa, \xi \in \Theta$, we derive that

$$\|G\|_{M_{\rho_t}} \geq (\tilde{C}(G))\rho_t(\|G\|_{\text{ber}}).$$

In a similar fashion, we can assert that

$$\|G\|_{M_{\rho_t}} \geq (\|G\|_{\text{ber}})\rho_t(\tilde{C}(G)).$$

By combining these two inequalities, we confidently arrive at the first inequality. For the second inequality, employing the Cauchy-Schwarz inequality leads us to

$$|\langle G\hat{k}_{\kappa}, \hat{k}_{\xi} \rangle|^2 \leq |\langle G\hat{k}_{\kappa}, G\hat{k}_{\kappa} \rangle| |\langle \hat{k}_{\xi}, \hat{k}_{\xi} \rangle| = |\langle G^*G\hat{k}_{\kappa}, \hat{k}_{\kappa} \rangle| \leq \text{ber}(G^*G) = \|G^*G\|_{\text{ber}}$$

and

$$|\langle G^* \hat{k}_\kappa, \hat{k}_\xi \rangle|^2 \leq |\langle G^* \hat{k}_\kappa, G^* \hat{k}_\kappa \rangle| |\langle \hat{k}_\xi, \hat{k}_\xi \rangle| = |\langle GG^* \hat{k}_\kappa, \hat{k}_\kappa \rangle| \leq \mathbf{ber}(GG^*) = \|GG^*\|_{\mathbf{ber}}.$$

By the monotonicity of ρ_t , we have

$$\begin{aligned} \|G\|_{\rho_t} &= \sup_{\kappa, \xi \in \Theta} \left\{ |\langle G \hat{k}_\kappa, \hat{k}_\xi \rangle| \rho_t |\langle G^* \hat{k}_\kappa, \hat{k}_\xi \rangle| \right\} \\ &\leq \min \left\{ \|G^* G\|_{\mathbf{ber}}^{\frac{1}{2}}, \|GG^*\|_{\mathbf{ber}}^{\frac{1}{2}} \right\}. \end{aligned} \quad \square$$

In the following result, we will explore some special cases of Theorem 10.

Corollary 11 *Let $G \in \mathcal{O}(\mathcal{K})$, $s \in [-1, 1]$ and $0 \leq t \leq 1$. Then*

$$\max \left\{ \sqrt[s]{\frac{(1-t)\|G\|_{\mathbf{ber}}^s + t\tilde{C}^s(G)}{2}}, \sqrt[s]{\frac{(1-t)\tilde{C}^s(G) + t\|G\|_{\mathbf{ber}}^s}{2}} \right\} \leq \|G\|_{M_{s,t}}.$$

In particular,

$$\max \left\{ \frac{(1-t)\|G\|_{\mathbf{ber}} + t\tilde{C}^r(G)}{2}, \frac{(1-t)\tilde{C}^r(G) + t\|G\|_{\mathbf{ber}}}{2} \right\} \leq \|G\|_{M_{\nabla_t}}.$$

and

$$\max \left\{ \frac{\|G\|_{\mathbf{ber}}^{(1-t)} \tilde{C}^t(G)}{2}, \frac{\tilde{C}^{(1-t)}(G) \|G\|_{\mathbf{ber}}^t}{2} \right\} \leq \|G\|_{M_{\sharp_t}}.$$

Proof Let ρ_t represent the interpolational paths of the power means $p_{s,t}$ for $s \in [-1, 1]$ and $0 \leq t \leq 1$, as established in Theorem 10. From this foundation, we derive the first inequality. Furthermore, upon considering the means ∇_t and $\sharp_t$ (where $0 \leq t \leq 1$) as outlined in Theorem 10, we are able to obtain the second and third inequalities, respectively. $\square$

Theorem 12 *Let $G \in \mathcal{O}(\mathcal{K})$, and let ρ be a mean dominated by the arithmetic mean ∇ . Additionally, let $0 \leq \alpha, \beta \leq 1$ such that $\alpha + \beta = 1$. Then*

$$\|G\|_{M_\rho} \leq \frac{1}{2} \left[\left\| \alpha |G|^2 + \beta |G^*|^2 \right\|_{\mathbf{ber}}^{\frac{1}{2}} + \left\| (1-\alpha) |G|^2 + (1-\beta) |G^*|^2 \right\|_{\mathbf{ber}}^{\frac{1}{2}} \right].$$

In particular,

$$\|G\|_{M_\rho}^2 \leq \frac{1}{2} \left\| |G|^2 + |G^*|^2 \right\|_{\mathbf{ber}}.$$

Proof Let $\kappa, \xi \in \Theta$. Then, applying Lemma 5, we have

$$\begin{aligned} &(|\langle G \hat{k}_\kappa, \hat{k}_\xi \rangle|) \rho (|\langle G^* \hat{k}_\kappa, \hat{k}_\xi \rangle|) \\ &\leq \left(\sqrt{\langle |G|^{2\alpha} \hat{k}_\kappa, \hat{k}_\kappa \rangle} \sqrt{\langle |G^*|^{2(1-\alpha)} \hat{k}_\xi, \hat{k}_\xi \rangle} \right) \rho \left(\sqrt{\langle |G^*|^{2\beta} \hat{k}_\kappa, \hat{k}_\kappa \rangle} \sqrt{\langle |G|^{2(1-\beta)} \hat{k}_\xi, \hat{k}_\xi \rangle} \right) \end{aligned}$$

$$\begin{aligned}
&\leq \left(\sqrt{\langle |G|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle^\alpha} \sqrt{\langle |G^*|^2 \hat{k}_\xi, \hat{k}_\xi \rangle^{(1-\alpha)}} \right) \rho \left(\sqrt{\langle |G^*|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle^\beta} \sqrt{\langle |G|^2 \hat{k}_\xi, \hat{k}_\xi \rangle^{(1-\beta)}} \right) \\
&\quad \text{(by Lemma 4)} \\
&\leq \frac{1}{2} \left(\sqrt{\langle |G|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle^\alpha} \sqrt{\langle |G^*|^2 \hat{k}_\xi, \hat{k}_\xi \rangle^{(1-\alpha)}} + \sqrt{\langle |G^*|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle^\beta} \sqrt{\langle |G|^2 \hat{k}_\xi, \hat{k}_\xi \rangle^{(1-\beta)}} \right) \\
&\leq \frac{1}{2} \left[\left(\alpha \langle |G|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle^{\frac{1}{2}} + (1-\alpha) \langle |G^*|^2 \hat{k}_\xi, \hat{k}_\xi \rangle^{\frac{1}{2}} \right) \right. \\
&\quad \left. + \left(\beta \langle |G^*|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle^{\frac{1}{2}} + (1-\beta) \langle |G|^2 \hat{k}_\xi, \hat{k}_\xi \rangle^{\frac{1}{2}} \right) \right] \\
&\quad \text{(by the inequality } u^\alpha v^{1-\alpha} \leq \alpha u + (1-\alpha)v \text{)} \\
&= \frac{1}{2} \left[\left(\alpha \langle |G|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle^{\frac{1}{2}} + \beta \langle |G^*|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle^{\frac{1}{2}} \right) \right. \\
&\quad \left. + \left((1-\alpha) \langle |G^*|^2 \hat{k}_\xi, \hat{k}_\xi \rangle^{\frac{1}{2}} + (1-\beta) \langle |G|^2 \hat{k}_\xi, \hat{k}_\xi \rangle^{\frac{1}{2}} \right) \right] \\
&\leq \frac{1}{2} \left[\left(\alpha \langle |G|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle + \beta \langle |G^*|^2 \hat{k}_\kappa, \hat{k}_\kappa \rangle \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left((1-\alpha) \langle |G^*|^2 \hat{k}_\xi, \hat{k}_\xi \rangle + (1-\beta) \langle |G|^2 \hat{k}_\xi, \hat{k}_\xi \rangle \right)^{\frac{1}{2}} \right] \\
&\quad \text{(by the concavity of } h(x) = \sqrt{x} \text{)} \\
&= \frac{1}{2} \left[\left(\langle (\alpha |G|^2 + \beta |G^*|^2) \hat{k}_\kappa, \hat{k}_\kappa \rangle \right)^{\frac{1}{2}} + \left(\langle ((1-\alpha) |G|^2 + (1-\beta) |G^*|^2) \hat{k}_\xi, \hat{k}_\xi \rangle \right)^{\frac{1}{2}} \right] \\
&\leq \frac{1}{2} \left[\left\| \alpha |G|^2 + \beta |G^*|^2 \right\|_{\text{ber}}^{\frac{1}{2}} + \left\| (1-\alpha) |G|^2 + (1-\beta) |G^*|^2 \right\|_{\text{ber}}^{\frac{1}{2}} \right],
\end{aligned}$$

whence

$$\begin{aligned}
&\frac{1}{2} \left[\left\| \alpha |G|^2 + \beta |G^*|^2 \right\|_{\text{ber}}^{\frac{1}{2}} + \left\| (1-\alpha) |G|^2 + (1-\beta) |G^*|^2 \right\|_{\text{ber}}^{\frac{1}{2}} \right] \\
&\geq \sup_{\kappa, \xi \in \Theta} \left\{ \left(\langle |G \hat{k}_\kappa, \hat{k}_\xi \rangle \right) \rho \left(\langle |G^* \hat{k}_\kappa, \hat{k}_\xi \rangle \right) \right\} \\
&= \|G\|_{M_\rho}.
\end{aligned}$$

Hence, we derive the first inequality. For the second inequality, setting $\alpha = \beta = \frac{1}{2}$ yields the desired result. $\square$

Theorem 13 Let $G \in \mathcal{O}(\mathcal{K})$ and let ρ be a mean that is dominated by the arithmetic mean ∇ . Consider two non-negative continuous functions f_1, f_2, g_1 , and g_2 defined on the interval $[0, \infty)$. It is established that $f_1(t)g_1(t) = t$ and $f_2(t)g_2(t) = t$ for all $t \geq 0$. Then

$$\|G\|_{M_\rho}^2 \leq \frac{1}{8} \left(\|f_1^2(|G|) + g_2^2(|G^*|)\|_{\text{ber}} + \|f_2^2(|G^*|) + g_1^2(|G|)\|_{\text{ber}} \right).$$

In particular,

$$\|G\|_{M_\rho} \leq \frac{1}{2} \sqrt{\| |G|^{2r} + |G^*|^{2(1-r)} \|_{\text{ber}}},$$

where $0 \leq r \leq 1$.

Proof Assume that $\kappa, \xi \in \Theta$. Then by using Lemma 5, we have

$$|\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle| \leq \sqrt{|\langle f_1^2(|G|)\hat{k}_\kappa, \hat{k}_\kappa \rangle|} \sqrt{|\langle g_1^2(|G^*|)\hat{k}_\xi, \hat{k}_\xi \rangle|}$$

and

$$|\langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle| \leq \sqrt{|\langle g_2^2(|G^*|)\hat{k}_\kappa, \hat{k}_\kappa \rangle|} \sqrt{|\langle f_2^2(|G|)\hat{k}_\xi, \hat{k}_\xi \rangle|}.$$

Therefore,

$$\begin{aligned} & \left(|\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle| \rho |\langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle| \right)^2 \\ & \leq \left(\sqrt{|\langle f_1^2(|G|)\hat{k}_\kappa, \hat{k}_\kappa \rangle|} \sqrt{|\langle g_1^2(|G^*|)\hat{k}_\xi, \hat{k}_\xi \rangle|} \rho \sqrt{|\langle g_2^2(|G^*|)\hat{k}_\kappa, \hat{k}_\kappa \rangle|} \sqrt{|\langle f_2^2(|G|)\hat{k}_\xi, \hat{k}_\xi \rangle|} \right)^2 \\ & \leq \frac{1}{4} \left[\sqrt{|\langle f_1^2(|G|)\hat{k}_\kappa, \hat{k}_\kappa \rangle|} \sqrt{|\langle g_1^2(|G^*|)\hat{k}_\xi, \hat{k}_\xi \rangle|} + \sqrt{|\langle g_2^2(|G^*|)\hat{k}_\kappa, \hat{k}_\kappa \rangle|} \sqrt{|\langle f_2^2(|G|)\hat{k}_\xi, \hat{k}_\xi \rangle|} \right]^2 \\ & \quad (\text{by } \rho \leq \nabla) \\ & \leq \frac{1}{4} \left(|\langle f_1^2(|G|)\hat{k}_\kappa, \hat{k}_\kappa \rangle| + |\langle g_2^2(|G^*|)\hat{k}_\kappa, \hat{k}_\kappa \rangle| \right) \left(|\langle g_1^2(|G^*|)\hat{k}_\xi, \hat{k}_\xi \rangle| + |\langle f_2^2(|G|)\hat{k}_\xi, \hat{k}_\xi \rangle| \right) \\ & \quad (\text{by the Cauchy-Schwarz inequality } \sqrt{ab} + \sqrt{cd} \leq \sqrt{a+c} \sqrt{b+d}) \\ & \leq \frac{1}{8} \left[|\langle (f_1^2(|G|) + g_2^2(|G^*|))\hat{k}_\kappa, \hat{k}_\kappa \rangle| + |\langle (g_1^2(|G^*|) + f_2^2(|G|))\hat{k}_\xi, \hat{k}_\xi \rangle| \right] \\ & \quad (\text{by the arithmetic-geometric mean inequality}) \\ & \leq \frac{1}{8} \left(\|f_1^2(|G|) + g_2^2(|G^*|)\|_{\text{ber}} + \|f_2^2(|G^*|) + g_1^2(|G|)\|_{\text{ber}} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \|G\|_{M_\rho}^2 &= \sup_{\kappa, \xi \in \Theta} \left\{ \left(|\langle G\hat{k}_\kappa, \hat{k}_\xi \rangle| \rho |\langle G^*\hat{k}_\kappa, \hat{k}_\xi \rangle| \right)^2 \right\} \\ &\leq \frac{1}{8} \left(\|f_1^2(|G|) + g_2^2(|G^*|)\|_{\text{ber}} + \|f_2^2(|G^*|) + g_1^2(|G|)\|_{\text{ber}} \right). \end{aligned}$$

We establish the first inequality confidently. For the special case, we choose $f_1(t) = g_1(t) = t^r$ and $f_2(t) = g_2(t) = t^{1-r}$ (with $0 \leq r \leq 1$), which effectively leads us to the desired result. $\square$

3 Some applications

In this section, we first explore our definition of the Volterra integration operator within the Hardy space $\mathcal{H}^2(\mathbb{Q})$.

Remark 14 Assume $\mathcal{H}^2(\mathbb{Q})$ is the Hardy space, which $\mathbb{Q} = \{w \in \mathbb{C} : |w| < 1\}$ and $V : \mathcal{H}^2(\mathbb{Q}) \rightarrow \mathcal{H}^2(\mathbb{Q})$ is the Volterra integration operator defined by $V(f) = \int_0^z f(t)dt$ for $f \in \mathcal{H}^2(\mathbb{Q})$. Then for every $k_\kappa(z) = \sum_{n=0}^{\infty} \bar{\kappa}^n z^n = (1 - \bar{\kappa}z)^{-1}$, we have $V(z^n) = \frac{z^{n+1}}{n+1}$ for $n = 0, 1, \dots$ and

$$|\langle V\hat{k}_\kappa(z), \hat{k}_\xi(z) \rangle| = \left| \left\langle V \left(\frac{(1 - |\kappa|^2)^{\frac{1}{2}}}{1 - \bar{\kappa}z} \right), \frac{(1 - |\xi|^2)^{\frac{1}{2}}}{1 - \bar{\xi}z} \right\rangle \right|$$

$$\begin{aligned}
&= (1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \left| \left\langle V \left(\sum_{n=0}^{\infty} \bar{\kappa}^n z^n \right), \sum_{n=0}^{\infty} \bar{\xi}^n z^n \right\rangle \right| \\
&= (1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \left| \left\langle \sum_{n=0}^{\infty} \bar{\kappa}^n \frac{z^{n+1}}{n+1}, \sum_{n=0}^{\infty} \bar{\xi}^n z^n \right\rangle \right| \\
&= (1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \left| \sum_{n=0}^{\infty} \frac{\bar{\kappa}^n \bar{\xi}^{n+1}}{n+1} \right|.
\end{aligned}$$

Similarly, $|\langle V^* \hat{k}_{\kappa}(z), \hat{k}_{\xi}(z) \rangle| = |\langle \hat{k}_{\kappa}(z), V \hat{k}_{\xi}(z) \rangle| = (1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \left| \sum_{n=0}^{\infty} \frac{\bar{\kappa}^{n+1} \bar{\xi}^n}{n+1} \right|$. Therefore,

$$\begin{aligned}
&\mathbf{ber}(V) \\
&= \sup_{\kappa \in \mathbb{Q}} \left\{ |\langle V \hat{k}_{\kappa}(z), \hat{k}_{\kappa}(z) \rangle| \right\} \\
&= \sup_{\kappa \in \mathbb{Q}} \left\{ \left(\frac{1 - |\kappa|^2}{|\bar{\kappa}|} \right) \sum_{n=0}^{\infty} \frac{|\kappa|^{2(n+1)}}{n+1} \right\} \\
&= \sup_{\kappa \in \mathbb{Q}} \left\{ \left(\frac{|\kappa|^2 - 1}{|\bar{\kappa}|} \right) \ln(1 - |\kappa|^2) \right\} \\
&\simeq 0.49084 \\
&\leq \|V\|_{M_{\rho_t}} \\
&= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ |\langle V \hat{k}_{\kappa}(z), \hat{k}_{\xi}(z) \rangle| \rho_t |\langle V^* \hat{k}_{\kappa}(z), \hat{k}_{\xi}(z) \rangle| \right\} \\
&= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ \left((1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \left| \sum_{n=0}^{\infty} \bar{\kappa}^n \bar{\xi}^{n+1} \right| \right) \right. \\
&\quad \left. \rho_t \left((1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \left| \sum_{n=0}^{\infty} \bar{\kappa}^{n+1} \bar{\xi}^n \right| \right) \right\} \\
&= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ (1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \left(\left| \sum_{n=0}^{\infty} \bar{\kappa}^n \bar{\xi}^{n+1} \right| \rho_t \left| \sum_{n=0}^{\infty} \bar{\kappa}^{n+1} \bar{\xi}^n \right| \right) \right\} \\
&\leq \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ (1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \left(\frac{\sum_{n=0}^{\infty} |\kappa \xi|^{n+1}}{|\kappa|} \rho_t \frac{\sum_{n=0}^{\infty} |\kappa \xi|^{n+1}}{|\xi|} \right) \right\} \\
&= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ (1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \left(\frac{-\ln(1 - |\kappa \xi|)}{|\kappa|} \rho_t \frac{-\ln(1 - |\kappa \xi|)}{|\xi|} \right) \right\} \\
&= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ -(1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \ln(1 - |\kappa \xi|) \left(\frac{1}{|\kappa|} \rho_t \frac{1}{|\xi|} \right) \right\} \\
&\leq \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ -(1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \ln(1 - |\kappa \xi|) \max \left\{ \frac{1}{|\kappa|}, \frac{1}{|\xi|} \right\} \right\} \simeq \\
&= 0.540261 \\
&\leq \|V\|_{\mathbf{ber}} \\
&= 1.
\end{aligned}$$

For the special case $\rho_t = \nabla_{\frac{2}{3}}$, by the standard calculus we have

$$\sup_{\kappa, \xi \in \mathbb{Q}} \left\{ -(1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \ln(1 - |\kappa \xi|) \left(\frac{1}{3|\kappa|} + \frac{2}{3|\xi|} \right) \right\} \simeq 0.495558.$$

Therefore,

$$\mathbf{ber}(V) \simeq 0.49084 \lesssim \|V\|_{M_{\nabla_{\frac{2}{3}}}} \lesssim 0.495558 \leq \|V\|_{\mathbf{ber}} = 1.$$

Moreover, for the special case $\rho_t = \sharp_{\frac{2}{3}}$, we have

$$\sup_{\kappa, \xi \in \mathbb{Q}} \left\{ -(1 - |\kappa|^2)^{\frac{1}{2}} (1 - |\xi|^2)^{\frac{1}{2}} \ln(1 - |\kappa \xi|) \left(\frac{1}{|\kappa|^{1/3} |\xi|^{2/3}} \right) \right\} \simeq 0.49489.$$

Hence,

$$\mathbf{ber}(V) \simeq 0.49084 \lesssim \|V\|_{M_{\sharp_{\frac{2}{3}}}} \lesssim 0.49489 \leq \|V\|_{\mathbf{ber}} = 1.$$

Thus, this example demonstrates that $\|\cdot\|_{M_{\rho_t}}$ serves as an estimate between the Berezin number and the Berezin norm on the Hardy space $\mathcal{H}^2(\mathbb{Q})$.

Remark 15 Stampfli [32] demonstrated that unitary operators on any Hilbert space $\mathcal{K}$ can be characterized as invertible contractions with contractive inverses. Specifically, this means that an operator $G \in \mathcal{O}(\mathcal{K})$ satisfies $\|G\| \leq 1$ and $\|G^{-1}\| \leq 1$. Sano and Uchiyama [31] proved that an invertible operator G on $\mathcal{K}$ is unitary if and only if $w(G) \leq 1$ and $w(G^{-1}) \leq 1$. However, it is not generally true that an invertible operator $G \in \mathcal{O}(\mathcal{K})$ is unitary if $\mathbf{ber}(G) \leq 1$ and $\mathbf{ber}(G^{-1}) \leq 1$. In [22], the authors established a characterization for unitary operators in terms of Berezin numbers, showing that an invertible operator $G \in \mathcal{O}(\mathcal{K})$ is unitary if and only if $\mathbf{ber}(GG^*) \leq 1$ and $\mathbf{ber}(G^{*-1}G^{-1}) \leq 1$. In [7], the authors proved a characterization of the unitary operator, replacing GG^* with G^*G . Consequently, using our definition and the inequality (1), we have an invertible operator $G \in \mathcal{O}(\mathcal{K})$ is unitary if and only if $\|GG^*\|_{M_{\rho_t}} \leq 1$ and $\|G^{*-1}G^{-1}\|_{M_{\rho_t}} \leq 1$, ($0 \leq t \leq 1$) if and only if $\|G^*G\|_{M_{\rho_t}} \leq 1$ and $\|G^{-1}G^{*-1}\|_{M_{\rho_t}} \leq 1$, ($0 \leq t \leq 1$).

In the following remark, we will explore our definition of a bounded weighted shift operator on the Bergman space.

Remark 16 Let $\mathbb{Q} = \{w \in \mathbb{C} : |w| < 1\}$ be the open unit disc and $Hol(\mathbb{Q})$ denotes the set of all holomorphic functions in $\mathbb{Q}$. The Bergman space on the unit disc $\mathbb{Q}$ is defined [29] by

$$A^2(\mathbb{Q}) = \left\{ f \in Hol(D) : \int_{\mathbb{Q}} |f(z)|^2 dV(z) < \infty \right\},$$

where dV is the normalized area measure on $\mathbb{Q}$. $A^2(\mathbb{Q})$ has the orthonormal basis $\{e_n\}_{n=0}^{\infty}$, where $e_n(z) = \sqrt{n+1}z^n$ for all $n \geq 0$. For $f, g \in A^2(\mathbb{Q})$ the inner product can also be expressed as

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \frac{1}{n+1} \lambda_n \bar{\mu}_n,$$

where $f(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ and $g(z) = \sum_{n=0}^{\infty} \mu_n z^n$. The space $A^2(\mathbb{Q})$ is a reproducing kernel Hilbert space and the reproducing kernel for $A^2(\mathbb{Q})$ is

$$k_w(z) = \frac{1}{(1 - \bar{w}z)^2} = \sum_{n=0}^{\infty} (n+1) \bar{w}^n z^n, \quad z, w \in \mathbb{Q}.$$

For the function $f(z) = \sum_{n=0}^{\infty} \lambda_n z^n$, which is an element of $A^2(\mathbb{Q})$, we consider the bounded weighted shift operator on $A^2(\mathbb{Q})$ defined as

$$T \left(\sum_{n=0}^{\infty} \sqrt{n+1} \lambda_n z^n \right) = \sum_{n=0}^{\infty} \alpha_{n+1} \lambda_n z^n,$$

where the weights $\{\alpha_n\}_{n=0}^{\infty}$ is a bounded sequence, see [2]. Let L be a positive real number such that $|\alpha_n| \leq L$ for all $n = 0, 1, \dots$. We have

$$\begin{aligned} |\langle T \hat{k}_\kappa(z), \hat{k}_\xi(z) \rangle| &= (1 - |\kappa|^2)(1 - |\xi|^2) \left| \left\langle V \left(\sum_{n=0}^{\infty} (n+1) \bar{\kappa}^n z^n \right), \sum_{n=0}^{\infty} (n+1) \bar{\xi}^n z^n \right\rangle \right| \\ &= (1 - |\kappa|^2)(1 - |\xi|^2) \left| \left\langle \sum_{n=0}^{\infty} (n+1) \alpha_{n+1} \bar{\kappa}^n z^{n+1}, \sum_{n=0}^{\infty} (n+1) \bar{\xi}^n z^n \right\rangle \right| \\ &= (1 - |\kappa|^2)(1 - |\xi|^2) \left| \sum_{n=0}^{\infty} (n+1) \alpha_{n+1} \bar{\kappa}^n \xi^{n+1} \right|. \end{aligned}$$

Similarly, $|\langle T^* \hat{k}_\kappa(z), \hat{k}_\xi(z) \rangle| = (1 - |\kappa|^2)(1 - |\xi|^2) \left| \sum_{n=0}^{\infty} (n+1) \overline{\alpha_{n+1}} \bar{\kappa}^{n+1} \xi^n \right|$. Hence,

$$\begin{aligned} &\|T\|_{M_{\rho_t}} \\ &= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ |\langle T \hat{k}_\kappa(z), \hat{k}_\xi(z) \rangle| \rho_t |\langle T^* \hat{k}_\kappa(z), \hat{k}_\xi(z) \rangle| \right\} \\ &= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ \left((1 - |\kappa|^2)(1 - |\xi|^2) \left| \sum_{n=0}^{\infty} (n+1) \alpha_{n+1} \bar{\kappa}^n \xi^{n+1} \right| \right) \right. \\ &\quad \left. \rho_t \left((1 - |\kappa|^2)(1 - |\xi|^2) \left| \sum_{n=0}^{\infty} (n+1) \overline{\alpha_{n+1}} \bar{\kappa}^{n+1} \xi^n \right| \right) \right\} \\ &\leq \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ \left((1 - |\kappa|^2)(1 - |\xi|^2) L \sum_{n=0}^{\infty} (n+1) |\bar{\kappa}^n \xi^{n+1}| \right) \right. \\ &\quad \left. \rho_t \left((1 - |\kappa|^2)(1 - |\xi|^2) L \sum_{n=0}^{\infty} (n+1) |\bar{\kappa}^{n+1} \xi^n| \right) \right\} \\ &= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ \left((1 - |\kappa|^2)(1 - |\xi|^2) L \frac{|\xi|}{(1 - |\kappa \xi|)^2} \right) \rho_t \left((1 - |\kappa|^2)(1 - |\xi|^2) L \frac{|\kappa|}{(1 - |\kappa \xi|)^2} \right) \right\} \\ &= \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ (1 - |\kappa|^2)(1 - |\xi|^2) \frac{L}{(1 - |\kappa \xi|)^2} (|\kappa| \rho_t |\xi|) \right\} \\ &\leq \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ (1 - |\kappa|^2)(1 - |\xi|^2) \frac{L}{(1 - |\kappa \xi|)^2} \max\{|\kappa|, |\xi|\} \right\} \end{aligned}$$

$$= L \sup_{\kappa, \xi \in \mathbb{Q}} \left\{ \frac{(1 - |\kappa|^2)(1 - |\xi|^2)}{(1 - |\kappa\xi|)^2} \max\{|\kappa|, |\xi|\} \right\}.$$

It follows from $\sup_{\kappa, \xi \in \mathbb{Q}} \left\{ \frac{(1 - |\kappa|^2)(1 - |\xi|^2)}{(1 - |\kappa\xi|)^2} \max\{|\kappa|, |\xi|\} \right\} \simeq 1$ that

$$\|T\|_{M_{\rho_t}} \leq L \quad \text{for } 0 \leq t \leq 1. \quad (2)$$

Note that if $\alpha_n = \alpha$ for all $n = 0, 1, \dots$, then $\mathbf{ber}(T) = |\alpha|$, see [2]. In this case, it follows from the inequalities (1) and (2) that $\|T\|_{M_{\rho_t}} = |\alpha|$ for $0 \leq t \leq 1$.

Acknowledgements

The author would like to sincerely thank the referees for useful comments and suggestions. The author would like to indicate special thanks and gratitude to Professor Miguel Martin Suarez and the University of Granada, Department of Mathematical analysis to pass his sabbatical.

Author contributions

M.B. wrote the main manuscript text. The author reviewed the manuscript.

Funding information

The author declares that there is no source of funding for this research.

Data availability

No datasets were generated or analysed during the current study.

Declarations

Competing interests

The authors declare no competing interests.

Received: 16 May 2025 Accepted: 1 October 2025 Published online: 06 November 2025

References

1. Aronzañ, N.: Theory of reproducing kernels. *Trans. Am. Math. Soc.* **68**, 337–404 (1950)
2. Augustine, A., Das, P.H., Bhunia, P., Shankar, P.: On the convexity of Berezin range and Berezin radius inequalities via a class of semi-norms. <https://doi.org/10.48550/arXiv.2507.22427>
3. Augustine, A., Garayev, M.T., Shankar, P.: Composition operators, convexity of their Berezin range and related questions. *Complex Anal. Oper. Theory* **17**(8), 126 (2023). 22 pp.
4. Bakherad, M.: Some Berezin number inequalities for operator matrices. *Czechoslov. Math. J.* **68**(143)(4), 997–1009 (2018)
5. Bakherad, M., Conde, C., Kittaneh, F.: A new family of semi-norms between the Berezin radius and the Berezin norm. *Acta Appl. Math.* **192**, 3 (2024). 18 pp.
6. Bakherad, M., Karaev, M.T.: Berezin number inequalities for Hilbert space operators. *Concr. Oper.* **6**, 33–43 (2019)
7. Bhunia, P., Garayev, M.T., Paul, K., Tapdigoglu, R.: Some new applications of Berezin symbols. *Complex Anal. Oper. Theory* **17**(6), 96 (2023). 15 pp.
8. Bhunia, P., Paul, K., Sen, A.: Inequalities involving Berezin norm and Berezin number. *Complex Anal. Oper. Theory* **17**(1), 7 (2023). 15 pp.
9. Bhunia, P., Sen, A., Paul, K.: Development of the Berezin number inequalities. *Acta Math. Sin. Engl. Ser.* **39**(7), 1219–1228 (2023)
10. Chien, F., Bakherad, M., Alomari, M.W.: Refined Berezin number inequalities via superquadratic and convex functions. *Filomat* **37**, 265–277 (2023)
11. Conde, C., Moradi, H.R., Sababheh, M.: A family of semi-norms between the numerical radius and the operator norm. *Result. Math.* **79**(1), 36 (2024). 17 pp.
12. Garayev, M.T.: On the invertibility of operators on a model space. *N.Y. J. Math.* **30**, 436–450 (2024)
13. Garayev, M.T., Guesba, M.: Refinements of some inequalities involving Berezin norms and Berezin number and related questions. *Ann. Univ. Ferrara, Sez. 7: Sci. Mat.* **70**(2), 381–403 (2024)
14. Garayev, M.T., Gürdal, M., Saltan, S.: Hardy type inequality for reproducing kernel Hilbert space operators and related problems. *Positivity* **21**(4), 1615–1623 (2017)
15. Guillemin, V.: Toeplitz operators in n -dimensions. *Integral Equ. Oper. Theory* **7**(2), 145–205 (1984)
16. Hajmohamadi, M., Lashkaripour, R., Bakherad, M.: Improvements of Berezin number inequalities. *Linear Multilinear Algebra* **68**(6), 1218–1229 (2020)
17. Karaev, M.T.: On the Berezin symbol. *Zap. Nauč. Semin. POMI* **270**, 80–89 (2000). (Russian); Translated from *Zap. Nauč. Semin. POMI* **115**(2), 2135–2140 (2003)
18. Karaev, M.T.: Berezin symbol and invertibility of operators on the functional Hilbert spaces. *J. Funct. Anal.* **238**(1), 181–192 (2006)

19. Karaev, M.T.: Reproducing kernels and Berezin symbols techniques in various questions of operator theory. *Complex Anal. Oper. Theory* **7**(4), 983–1018 (2013)
20. Karaev, M.T., Gürdal, M.: On the Berezin symbols and Toeplitz operators. *Extr. Math.* **25**(1), 83–102 (2010)
21. Karaev, M.T., Gürdal, M., Huban, M.: Reproducing kernels, Engliš algebras and some applications. *Stud. Math.* **232**(2), 113–141 (2016)
22. Karaev, M.T., Iskenderov, N.: Berezin number of operators and related questions. *Methods Funct. Anal. Topol.* **19**, 68–72 (2013)
23. Karaev, M.T., Saltan, S.: Some results on Berezin symbols. *Complex Var. Theory Appl.* **50**(3), 185–193 (2005)
24. Kittaneh, F.: Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24**(2), 283–293 (1988)
25. Majee, S., Maji, A., Manna, A.: Numerical radius and Berezin number inequality. *J. Math. Anal. Appl.* **517**(1), 126566 (2023). 20 pp.
26. Mond, B., Pecaric, J.E.: Convex inequalities in Hilbert space. *Houst. J. Math.* **19**, 405–420 (1993)
27. Mond, B., Pecaric, J.E.: Classical inequalities for matrix functions. *Util. Math.* **46**, 155–166 (1994)
28. Nayak, R.K., Bhunia, P.: A new norm on the space of reproducing kernel Hilbert space operators and Berezin number inequalities. *Complex Anal. Oper. Theory* **19**, 83 (2025)
29. Paulsen, V.I., Raghupathi, M.: *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*. Cambridge University Press, Cambridge (2016)
30. Pecaric, J., Furuta, T., Hot, J.M., Seo, Y.: *Mond-Pecaric Method in Operator Inequalities*. Element, Zagreb (2005)
31. Sano, T., Uchiyama, A.: Numerical radius and unitarity. *Acta Sci. Math.* **76**, 581–584 (2010)
32. Stampfli, J.G.: Minimal range theorems for operators with thin spectra. *Pac. J. Math.* **23**, 601–612 (1967)
33. Tapdigoglu, R., Altwaijry, N., Garayev, M.T.: New inequalities via Berezin symbols and related questions. *Korean J. Math.* **31**(1), 109–120 (2023)
34. Yamanci, U., Garayev, M.T.: Some results related to the Berezin number inequalities. *Turk. J. Math.* **43**, 1940–1952 (2019)
35. Zhu, K.: *Operator Theory in Function Spaces*, 2nd. edn. Dekker, New York (2007)

Publisher's note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)