اپتیک فوریه

Fourier Optics

Optics- 4th Edition

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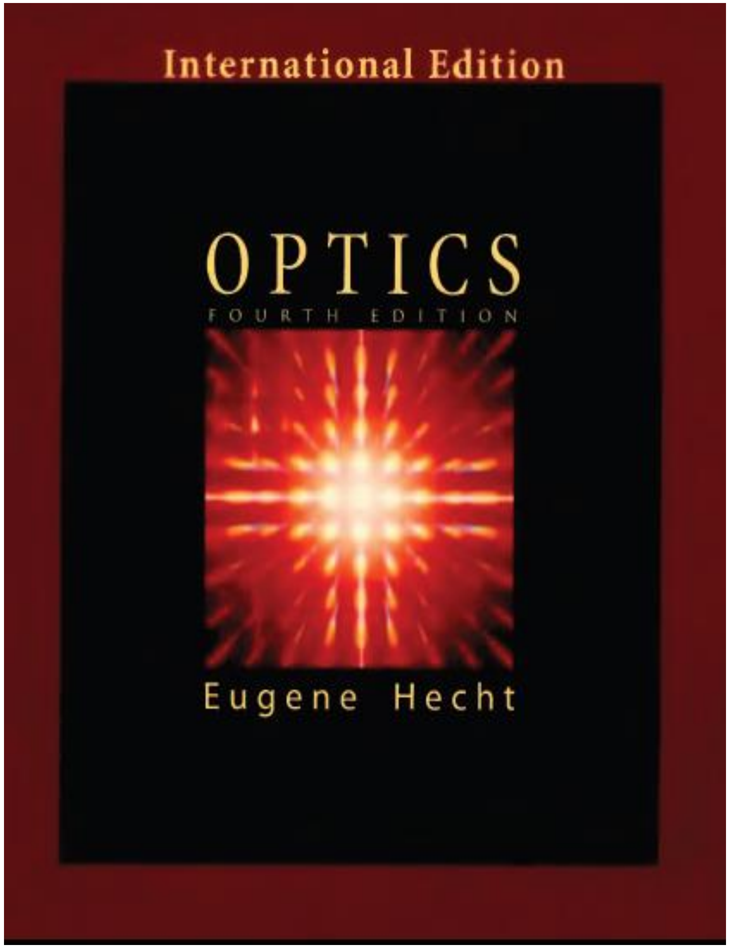
References

- Main references
  - **Textbook:**

Eugenem Hecht,

Optics- 4th Edition

Addison Wesly, 2002



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یوجن هشت، [مترجم] محمود دیانی،

موسسه علمی فرهنگی نص، ۱۳۸۵

11.1 Introduction

$$\psi(x) = A \sin kx = A \sin 2\pi x/\lambda = A \sin\varphi$$

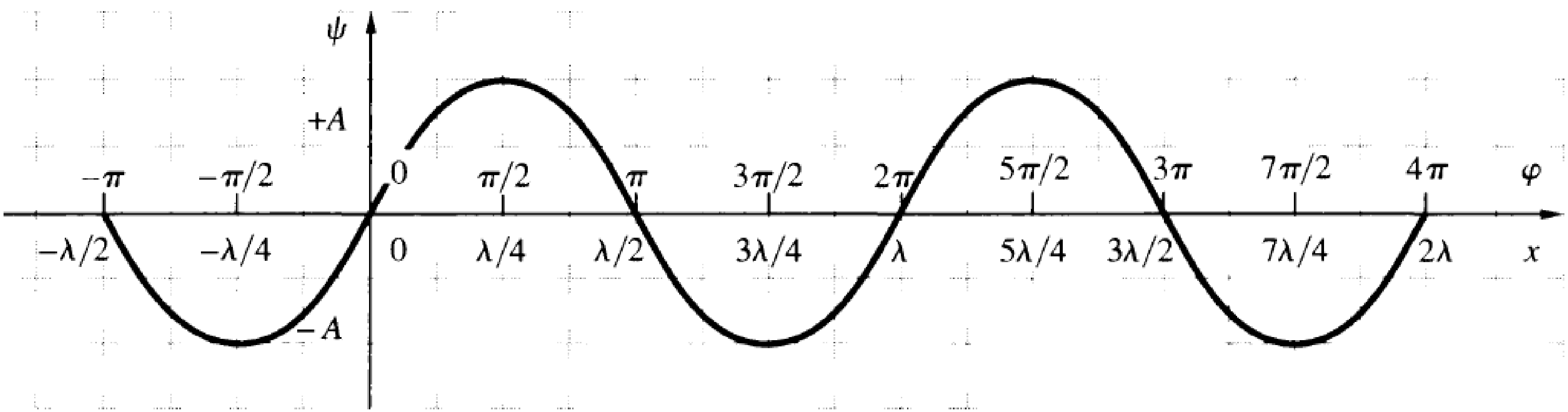


Figure 2.6 A harmonic function, which serves as the profile of a harmonic wave. One wavelength corresponds to a change in phase φ of 2π rad.

11.2 Fourier Transforms

11.2.1 One-Dimensional Transforms

$$f(x) = \frac{1}{\pi}\left[\int_0^{\infty} A(k)\cos kx\, dk + \int_0^{\infty} B(k)\sin kx\, dk\right] \quad [7.56]$$

$A(k)$ and $B(k)$, are the *Fourier cosine and sine transforms of* $f(x)$ given by

$$A(k) = \int_{-\infty}^{+\infty} f(x')\cos kx'\, dx'$$

and

$$B(k) = \int_{-\infty}^{+\infty} f(x')\sin kx'\, dx' \quad [7.57]$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \cos kx \int_{-\infty}^{+\infty} f(x') \cos kx'\, dx'\, dk$$

$$+ \frac{1}{\pi}\int_0^{\infty} \sin kx \int_{-\infty}^{+\infty} f(x') \sin kx'\, dx'\, dk$$

But since $\cos k(x' - x) = \cos kx \cos kx' + \sin kx \sin kx'$, this can be rewritten as

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{+\infty} f(x') \cos k(x'-x)\, dx'\right] dk \qquad (11.1)$$

The quantity in the square brackets is an even function of k, and therefore changing the limits on the outer integral leads to

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x') \cos k(x'-x)\, dx'\right] dk \qquad (11.2)$$

$$\frac{i}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x')\sin \boldsymbol{k}(x'-x)\,dx'\right]d\boldsymbol{k} = 0$$

because the factor in brackets is an odd function of $\boldsymbol{k}$. Adding these last two expressions yields the complex* form of the Fourier integral,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x')e^{i\boldsymbol{k}x'}\,dx'\right]e^{-i\boldsymbol{k}x}\,d\boldsymbol{k} \quad (11.3)$$

Thus we can write

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\boldsymbol{k})e^{-i\boldsymbol{k}x}\,d\boldsymbol{k} \quad (11.4)$$

provided that

$$F(\boldsymbol{k}) = \int_{-\infty}^{+\infty} f(x)e^{i\boldsymbol{k}x}\,dx \quad (11.5)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(k)e^{-ikx}\, dk \qquad (11.4)$$

provided that

$$F(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}\, dx \qquad (11.5)$$

$$\boxed{F(k) = \mathscr{F}\{f(x)\} \qquad (11.6)}$$

$$F(k) = A(k) + iB(k) \qquad (11.7a)$$

$$F(k) = |F(k)|e^{i\phi(k)} \qquad (11.7b)$$

inverse Fourier transform of $F(k)$, or symbolically

$$f(x) = \mathcal{F}^{-1}\{F(k)\} = \mathcal{F}^{-1}\{\mathcal{F}\{f(x)\}\} \tag{11.8}$$

$$\boxed{\kappa = 1/\lambda = k/2\pi}$$

$$\mathcal{F}\{F(k)\} = 2\pi f(-x) \quad \text{while} \quad \mathcal{F}^{-1}\{F(k)\} = f(x)$$

we would merely have to replace x by t and then k, the angular spatial frequency, by ω, the angular temporal frequency, in

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{-i\omega t}\,d\omega \tag{11.9}$$

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{i\omega t}\,dt \tag{11.10}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(k)e^{-ikx}\,dk$$

$$F(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}\,dx$$

$$f_3(x) = f_1(x) + f_2(x)$$

$$F_3(k) = F_1(k) + F_2(k)$$

$$\boxed{F(k) = \mathscr{F}\{f(x)\}}$$

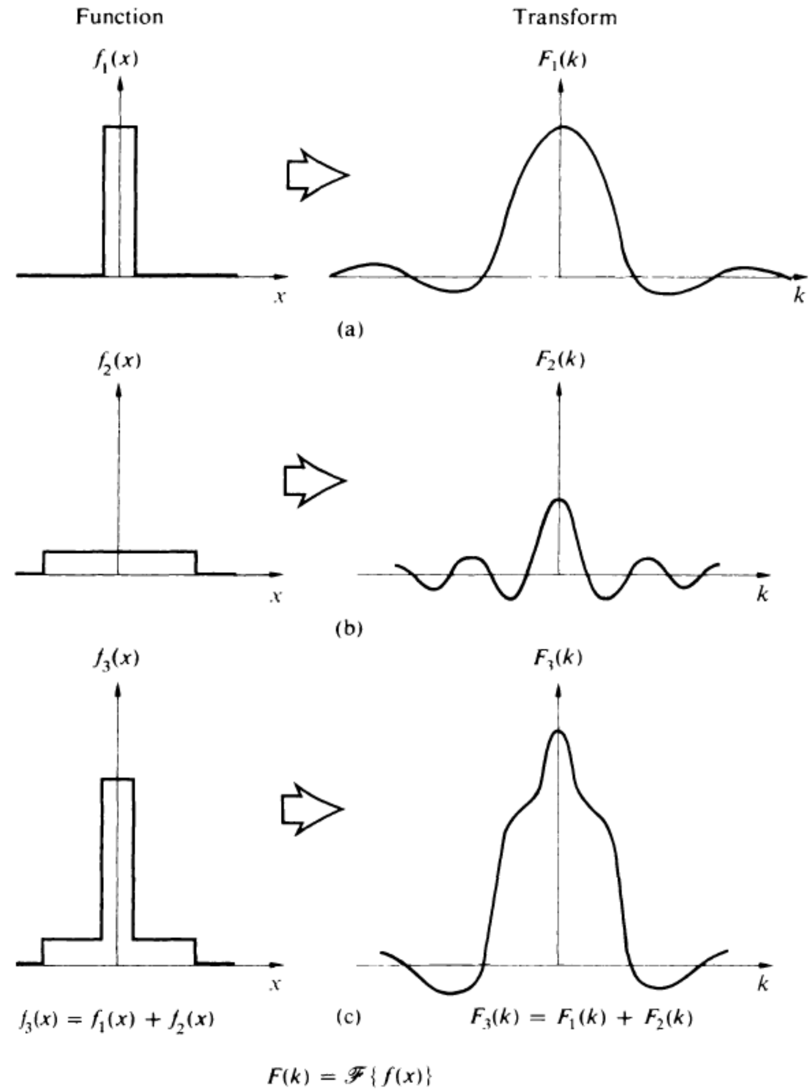


Figure 11.1 A composite function and its Fourier transform.

Transform of the Gaussian Function

$$\boxed{f(x) = Ce^{-ax^2}}$$

where $C = \sqrt{a/\pi}$ and a is a constant

$$F(\mathbf{k}) = \int_{-\infty}^{+\infty} (Ce^{-ax^2})e^{i\mathbf{k}x}\,dx$$

On completing the square, the exponent, $-ax^2 + i\mathbf{k}x$, becomes $-(x\sqrt{a} - i\mathbf{k}/2\sqrt{a})^2 - \mathbf{k}^2/4a$, and letting $x\sqrt{a} - i\mathbf{k}/2\sqrt{a} = \beta$ yields

$$F(\mathbf{k}) = \frac{C}{\sqrt{a}} e^{-\mathbf{k}^2/4a} \int_{-\infty}^{+\infty} e^{-\beta^2}\,d\beta$$

The definite integral can be found in tables and equals $\sqrt{\pi}$; hence

$$F(\mathbf{k}) = e^{-\mathbf{k}^2/4a} \tag{11.12}$$

$$\boxed{\sigma_x \sigma_k = 1}$$

$$\boxed{\sigma_x = 1/\sqrt{2a}}$$

$$\sigma_k = \sqrt{2a}$$

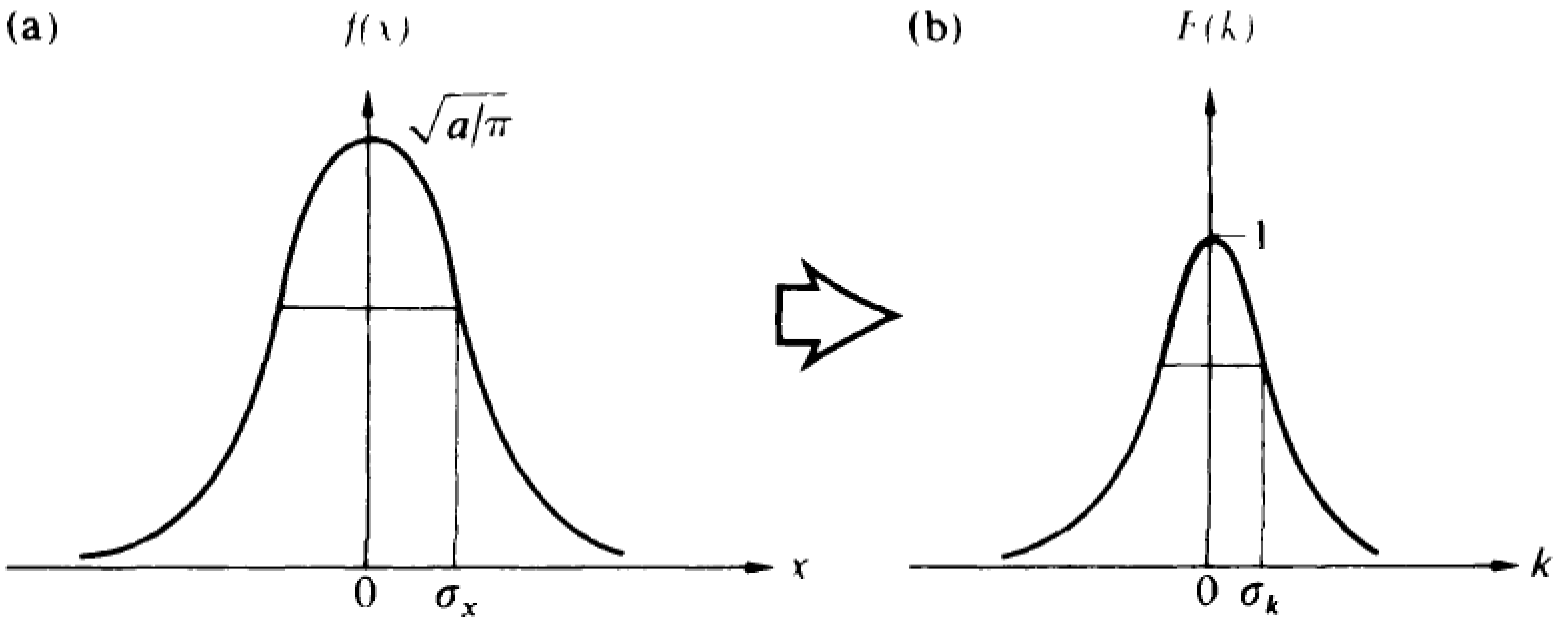


Figure 11.2 A Gaussian and its Fourier transform.

11.2.2 Two-Dimensional Transforms

$$f(x,y) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} F(k_x,k_y)e^{-i(k_x x + k_y y)} dk_x\, dk_y \qquad (11.13)$$

and

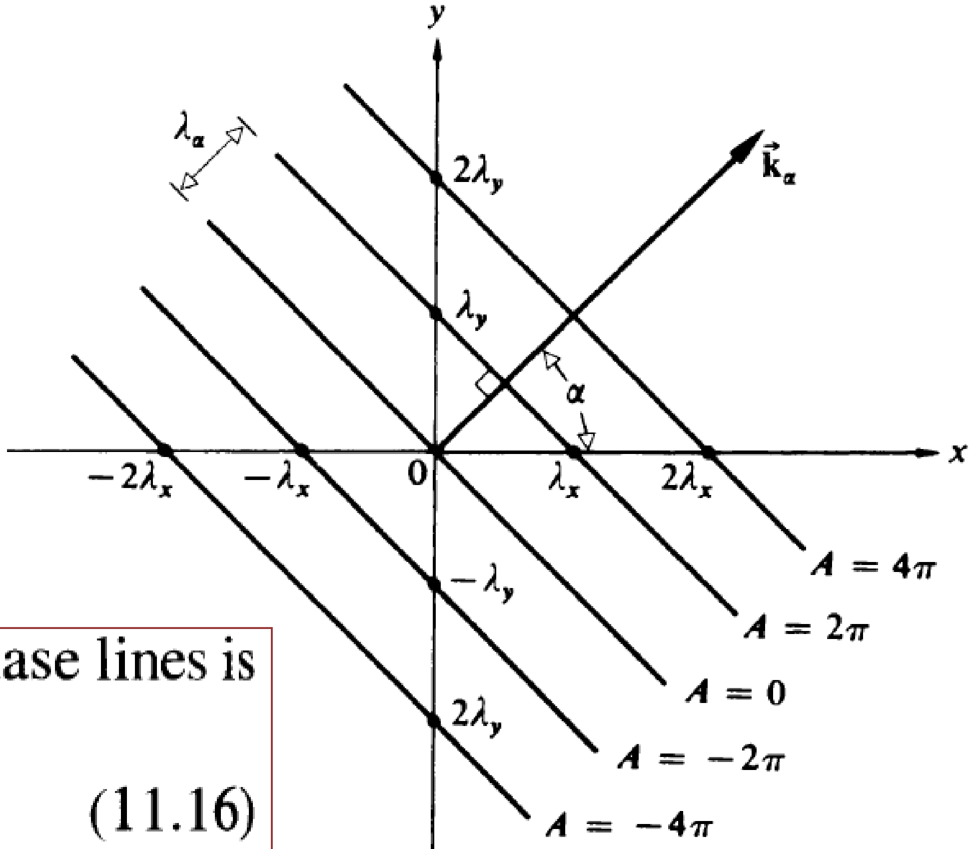
$$F(k_x,k_y) = \iint_{-\infty}^{+\infty} f(x,y)e^{i(k_x x + k_y y)}\, dx\, dy \qquad (11.14)$$

$$k_x x + k_y y = \text{constant} = A$$

$$y = -\frac{k_x}{k_y}x + \frac{A}{k_y} \qquad (11.15)$$

$$k_x x + k_y y = \text{constant} = A$$

$$y = -\frac{k_x}{k_y}x + \frac{A}{k_y}$$



The orientation of the constant phase lines is

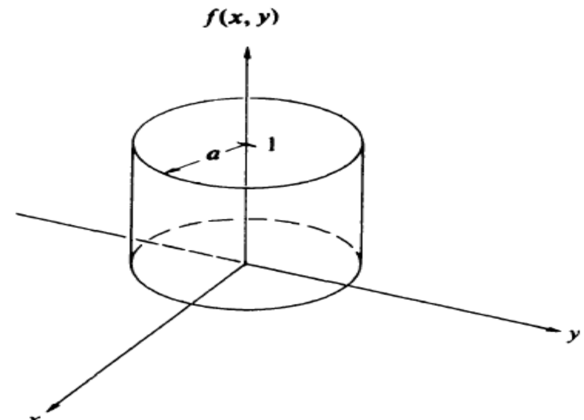
$$\alpha = \tan^{-1}\frac{k_y}{k_x} = \tan^{-1}\frac{\lambda_x}{\lambda_y} \qquad (11.16)$$

$$\lambda_\alpha = \frac{1}{\sqrt{\lambda_x^{-2} + \lambda_y^{-2}}}$$

$$k_\alpha = \sqrt{k_x^2 + k_y^2}$$

Transform of the Cylinder Function

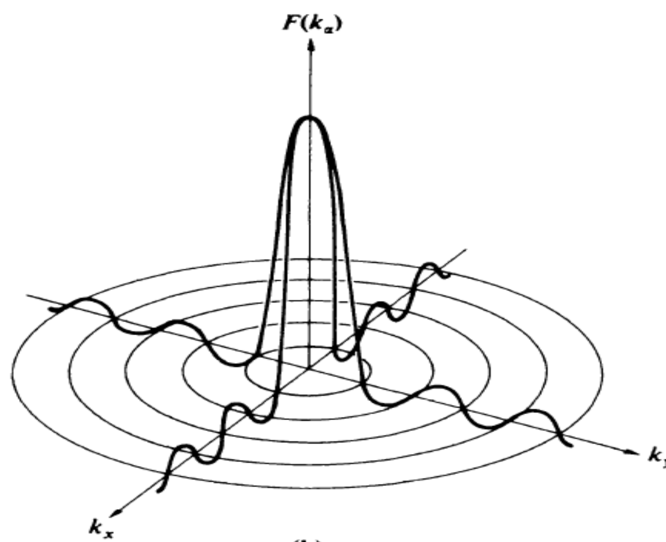
$$f(x,y) = \begin{cases} 1 & \sqrt{x^2+y^2} \le a \\ 0 & \sqrt{x^2+y^2} > a \end{cases} \qquad (11.19)$$



(a)

$$k_x = k_\alpha \cos\alpha$$
$$k_y = k_\alpha \sin\alpha$$
$$x = r\cos\theta \qquad (11.20)$$
$$y = r\sin\theta$$

$$dx\,dy = r\,dr\,d\theta.$$



(b)

$$F(k_\alpha,\alpha) = \int_{r=0}^{a}\left[\int_{\theta=0}^{2\pi} e^{ik_\alpha r\cos(\theta-\alpha)}\,d\theta\right]r\,dr \qquad (11.21)$$

$$F(k_\alpha) = \int_0^a \left[\int_0^{2\pi} e^{ik_\alpha r\cos\theta}\, d\theta \right] r\, dr \qquad (11.22)$$

$$F(k_\alpha) = 2\pi \int_0^a J_0(k_\alpha r) r\, dr \qquad (11.23)$$

$$\boxed{\frac{1}{k_\alpha^2} \int_{w=0}^{k_\alpha a} J_0(w) w\, dw \qquad (11.24)}$$

$$\boxed{\begin{aligned} F(k_\alpha) &= \frac{2\pi}{k_\alpha^2} k_\alpha a J_1(k_\alpha a) \\ F(k_\alpha) &= 2\pi a^2 \left[\frac{J_1(k_\alpha a)}{k_\alpha a} \right] \qquad (11.25) \end{aligned}}$$

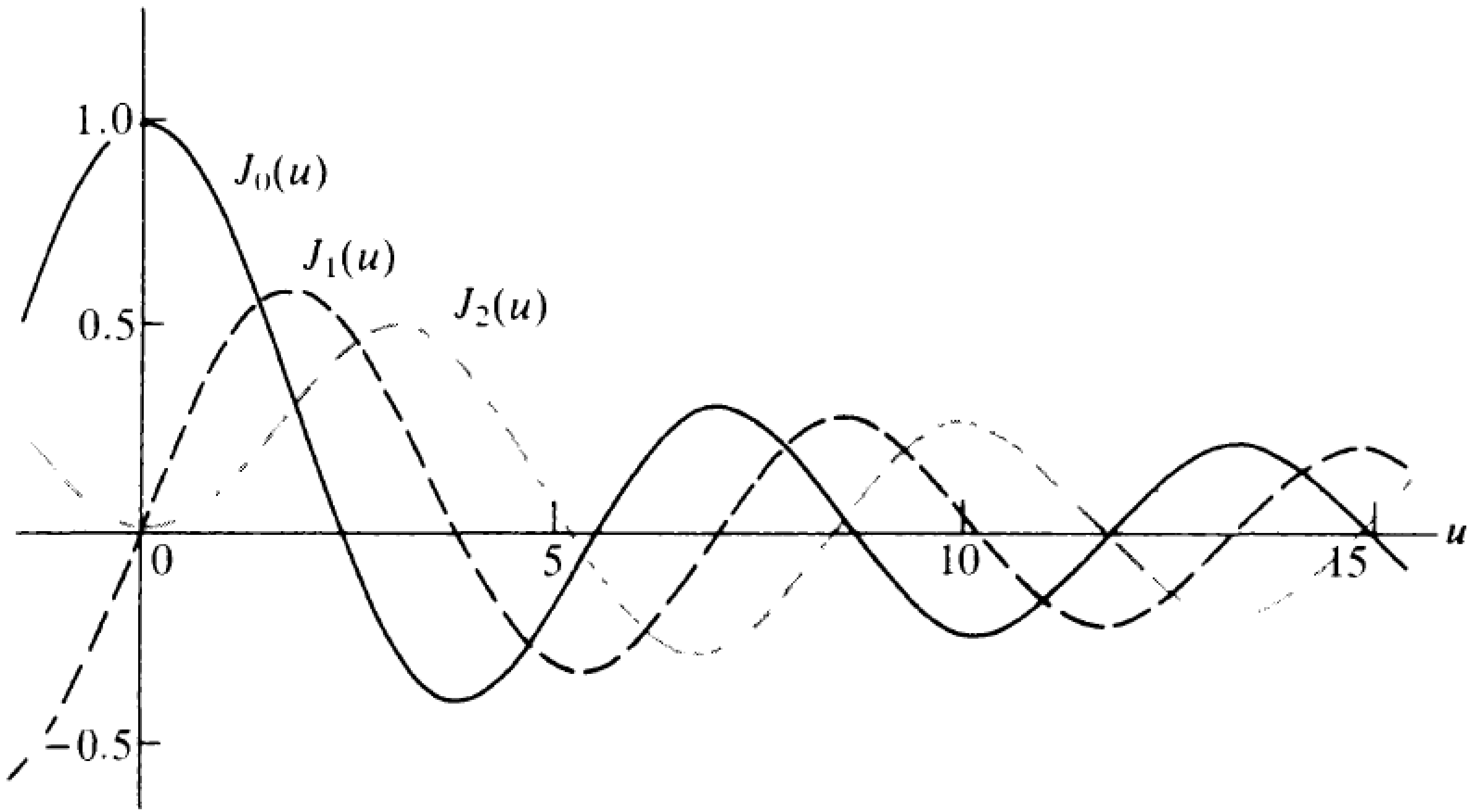


Figure 10.22 Bessel functions.

The Lens as a Fourier Transformer

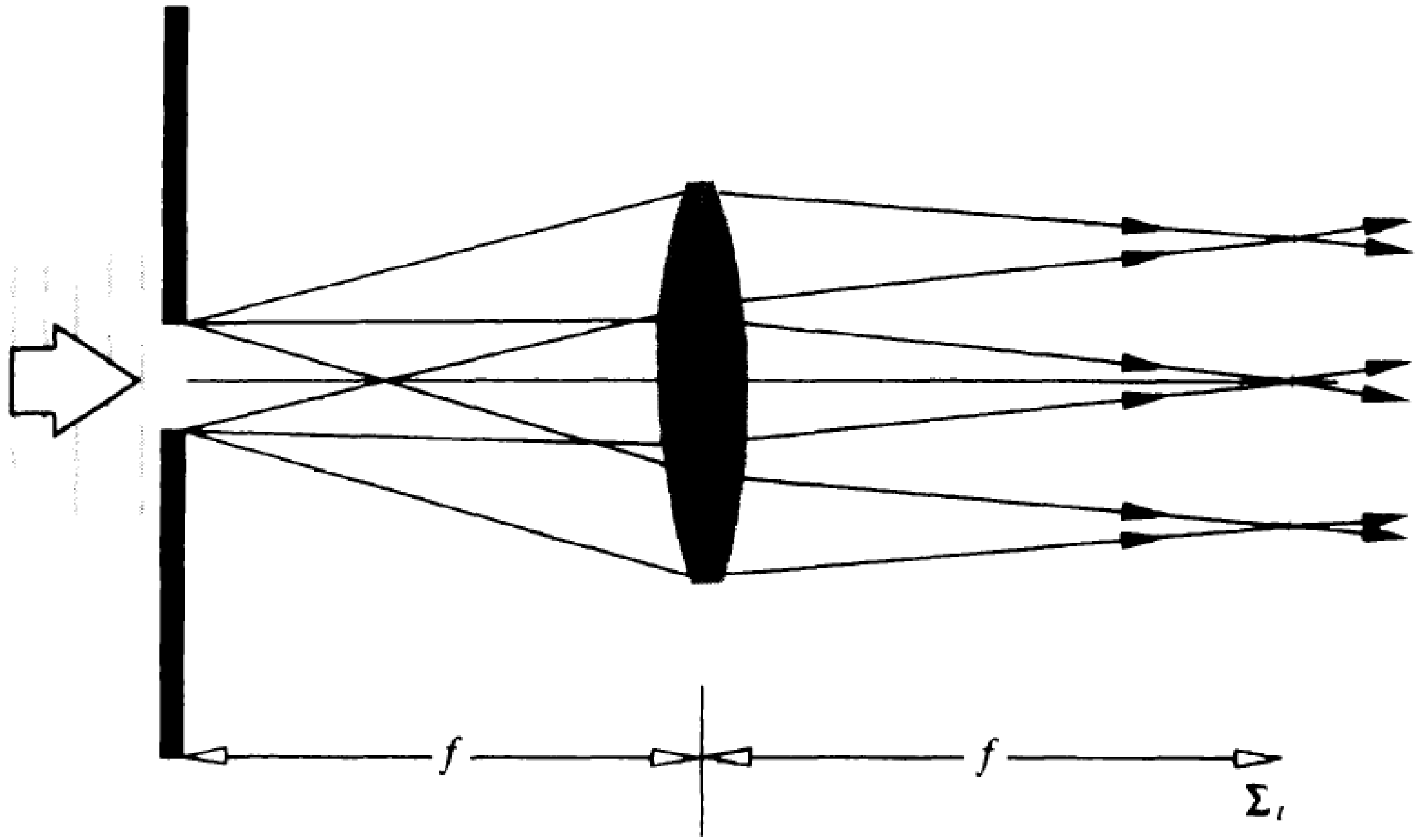
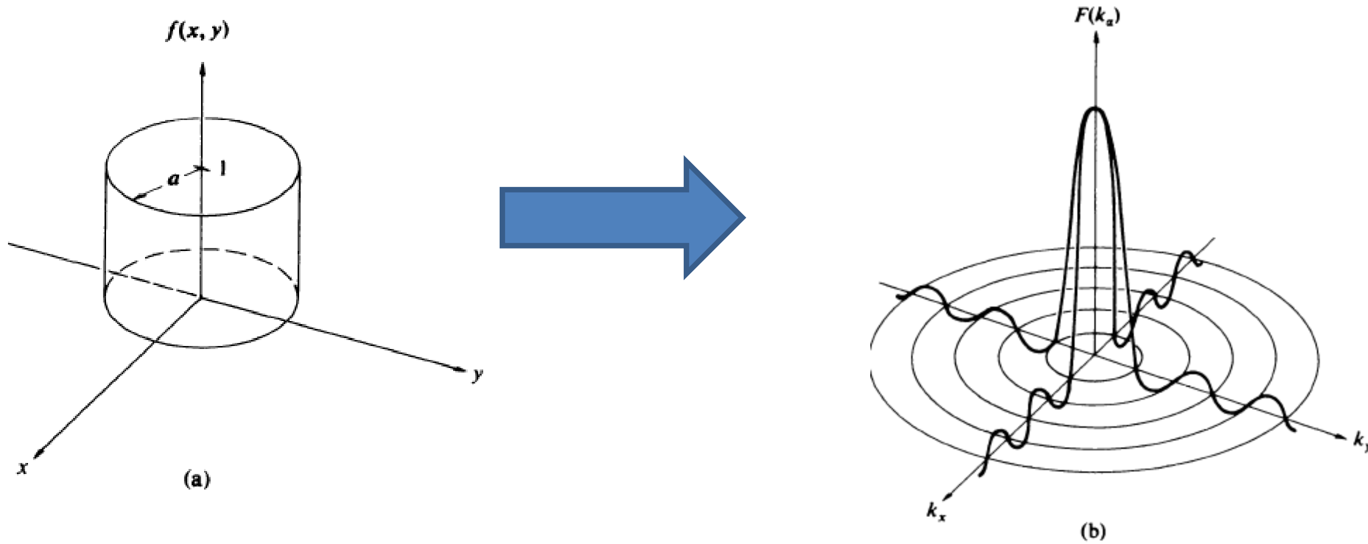


Figure 11.5 The light diffracted by a transparency at the front (or object) focal point of a lens converges to form the far-field diffraction pattern at the back (or image) focal point of the lens.

The Lens as a Fourier Transformer



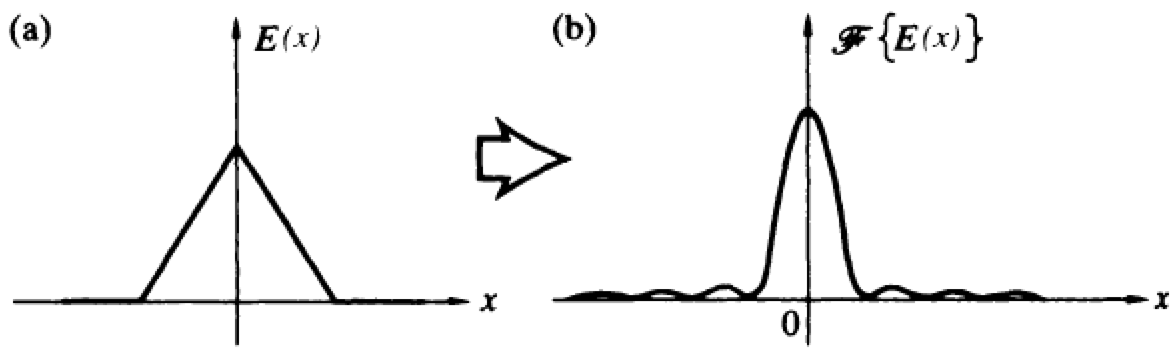


Figure 11.6 The transform of the triangle function is the sinc^2 function.

11.2.3 The Dirac Delta Function

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \qquad (11.26)$$

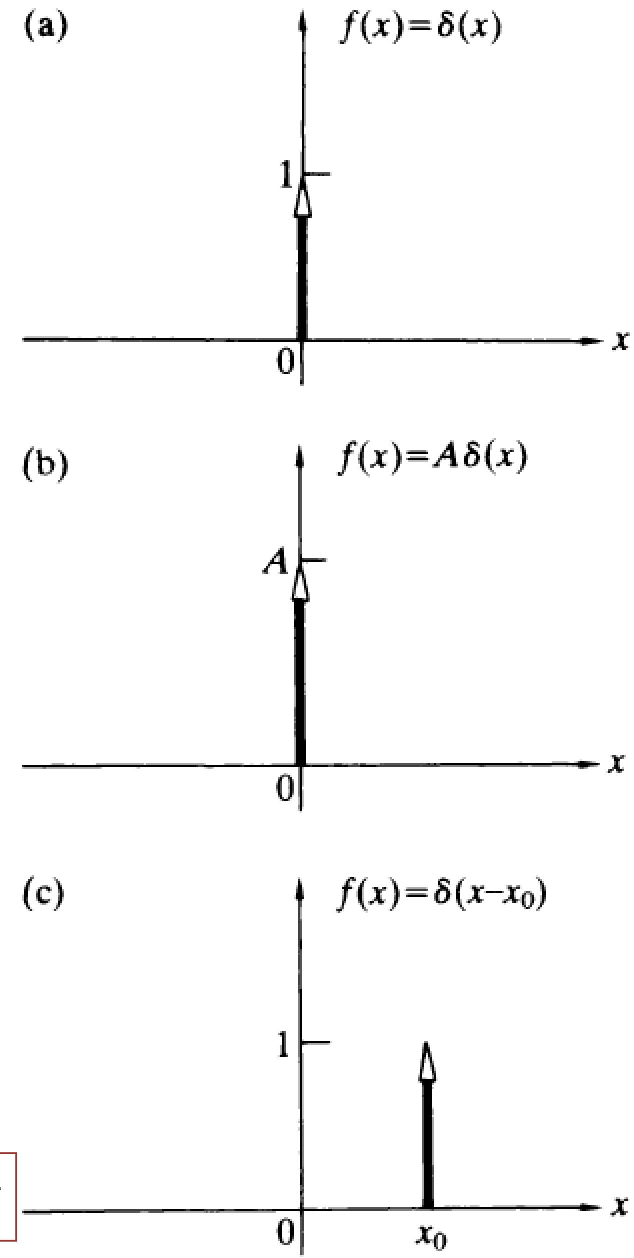
$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1 \qquad (11.27)$$

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = f(0) \int_{-\gamma}^{+\gamma} \delta(x)\, dx$$

$$\boxed{\int_{-\gamma}^{+\gamma} \delta(x)\, dx = 1}$$

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = f(0) \qquad (11.28)$$

sifting property



$$\delta(x - x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases} \qquad (11.29)$$

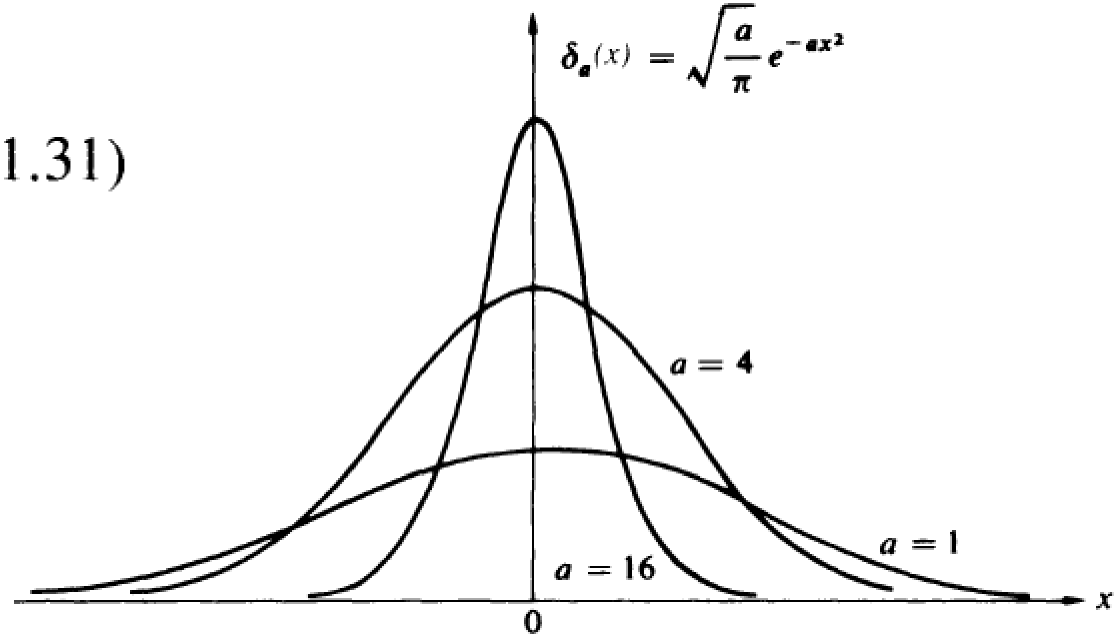
$$\int_{-\infty}^{+\infty} \delta(x - x_0) f(x)\, dx = \int_{-\infty}^{+\infty} \delta(x') g(x')\, dx' = g(0)$$

and since $g(0) = f(x_0)$,

$$\int_{-\infty}^{+\infty} \delta(x - x_0) f(x)\, dx = f(x_0) \qquad (11.30)$$

sequence of Gaussians

$$\delta_a(x) = \sqrt{\frac{a}{\pi}}\, e^{-ax^2} \qquad (11.31)$$



$$\delta_a(x) = \frac{a}{\pi} \operatorname{sinc}(ax) \qquad (11.32)$$

$$\lim_{a \to \infty} \int_{-\infty}^{+\infty} \delta_a(x) f(x)\, dx = f(0) \qquad (11.33)$$

delta sequences.

two dimensions

$$\delta(x,y) = \begin{cases} \infty & x = y = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (11.34)$$

$$\iint_{-\infty}^{+\infty} \delta(x,y)\, dx\, dy = 1 \qquad (11.35)$$

$$\iint_{-\infty}^{+\infty} f(x,y)\delta(x - x_0)\delta(y - y_0)\, dx\, dy = f(x_0,y_0) \qquad (11.36)$$

$$f(x) = \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik(x-x')}\, dk \right] f(x')\, dx'$$

$$f(x) = \int_{-\infty}^{+\infty} \delta(x - x') f(x')\, dx' \qquad (11.37)$$

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik(x-x')} dk \qquad (11.38)$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} dk \qquad (11.39)$$

$$\boxed{\delta(x) = \mathscr{F}^{-1}\{1\} \text{ and so } \mathscr{F}\{\delta(x)\} = 1}$$

Displacements and Phase Shifts

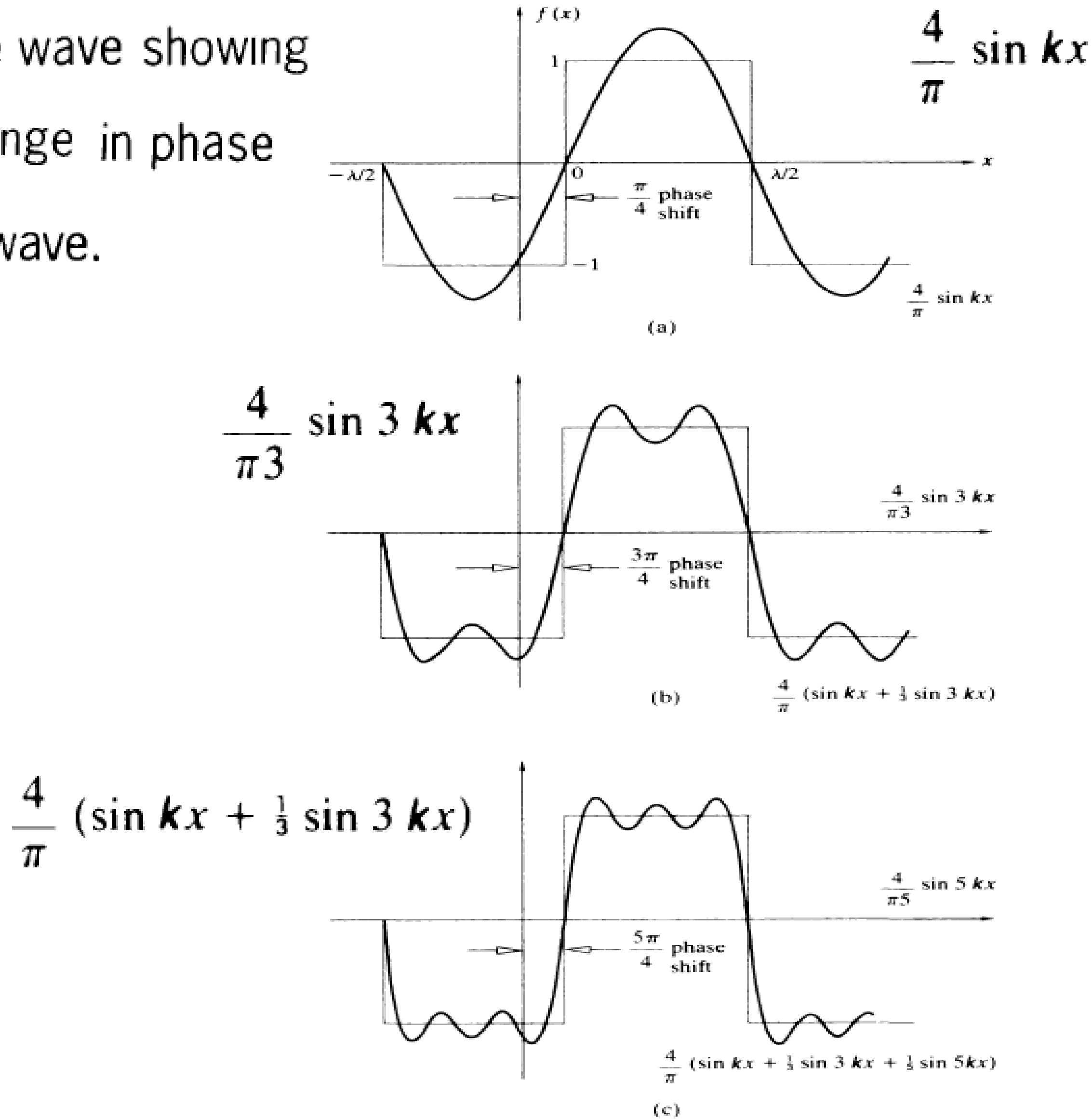
If the δ-spike is shifted off $x = 0$ to, say, $x = x_0$, its transform will change phase but not amplitude—that remains equal to one.

$$\mathscr{F}\{\delta(x - x_0)\} = \int_{-\infty}^{+\infty} \delta(x - x_0) e^{ikx}\, dx$$

$$\mathscr{F}\{\delta(x - x_0)\} = e^{ikx_0} \qquad (11.40)$$

All of this is quite general in its applicability, and we observe that ***the Fourier transform of a function that is displaced in space (or time) is the transform of the undisplaced function multiplied by an exponential that is linear in phase***

Figure 11.9 A shifted square wave showing the corresponding change in phase for each component wave.



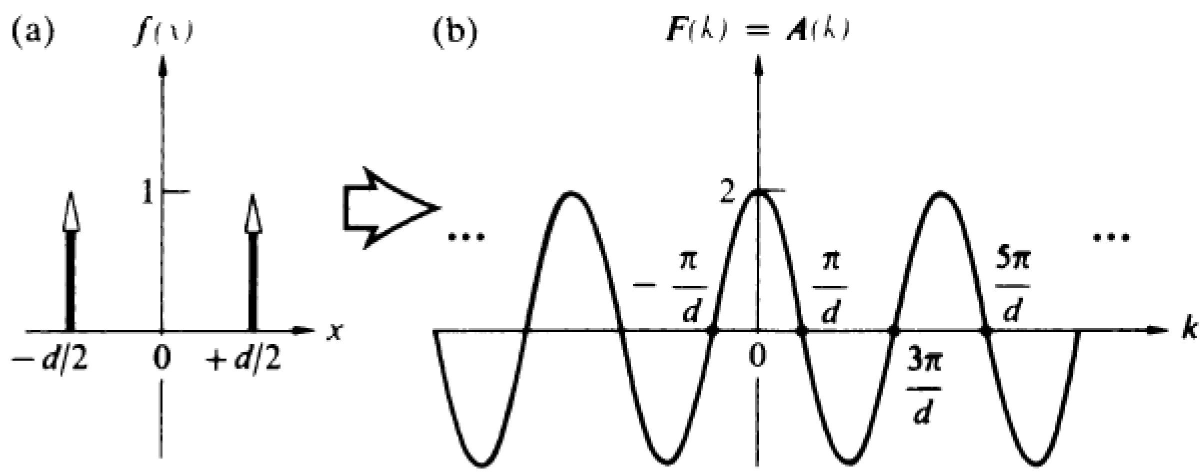
Sines and Cosines

Figure 11.10 Two delta functions and their cosine-function transform.

$\boxed{comb(x)}$

$$f(x) = \sum_j \delta(x - x_j) \quad (11.41)$$

$$\mathscr{F}\{f(x)\} = \sum_j e^{ikx_j} \quad (11.42)$$



$$f(x) = \delta[x - (+d/2)] + \delta[x - (-d/2)]$$

$$\mathscr{F}\{f(x)\} = e^{ikd/2} + e^{-ikd/2}$$

$$\mathscr{F}\{f(x)\} = 2\cos(kd/2) \quad (11.43)$$

Figure 11.11 Two delta functions and their sine-function transform

$$f(x) = \delta[x - (+d/2)] - \delta[x - (-d/2)]$$

$$\mathscr{F}\{f(x)\} = e^{ikd/2} - e^{ikd/2} = 2i \sin(kd/2) \quad (11.44)$$

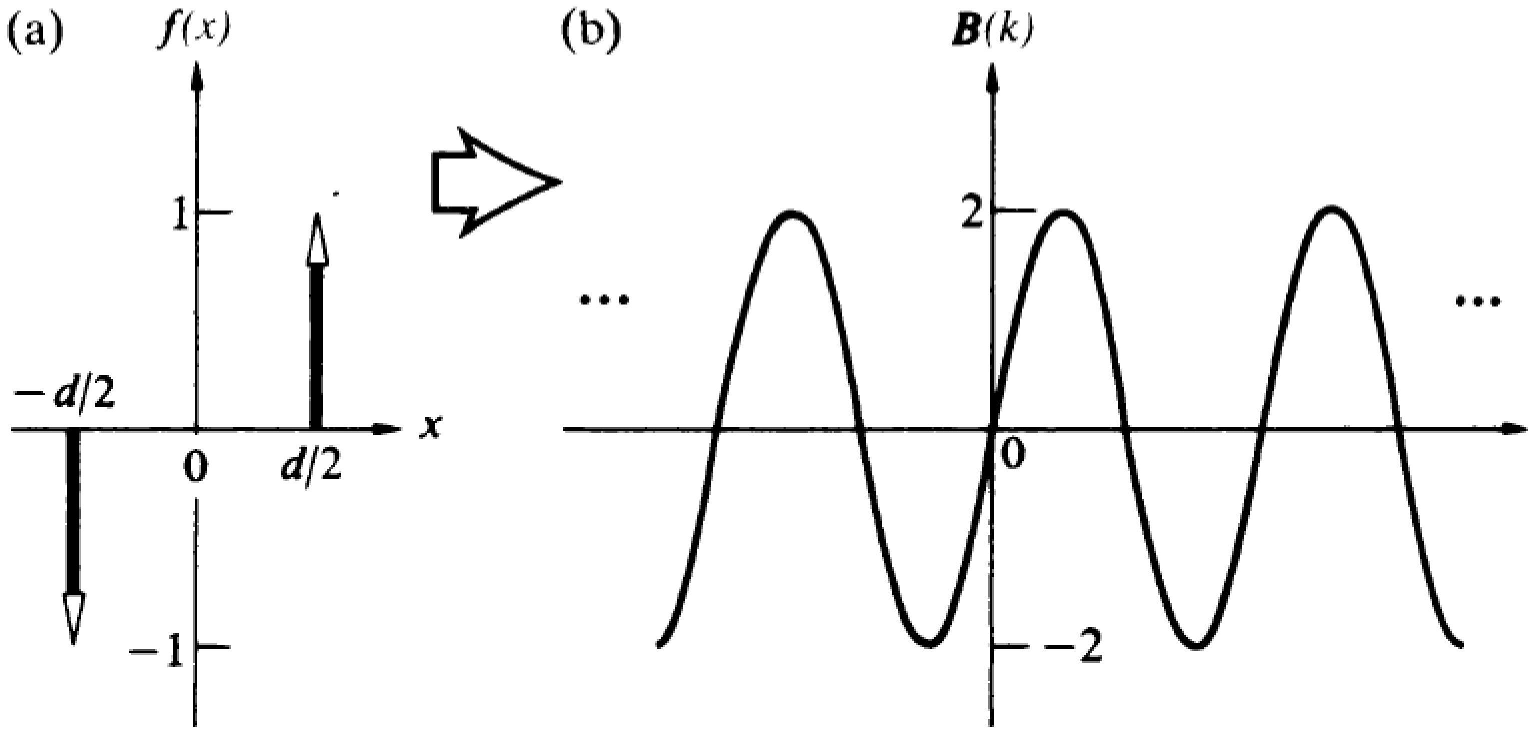
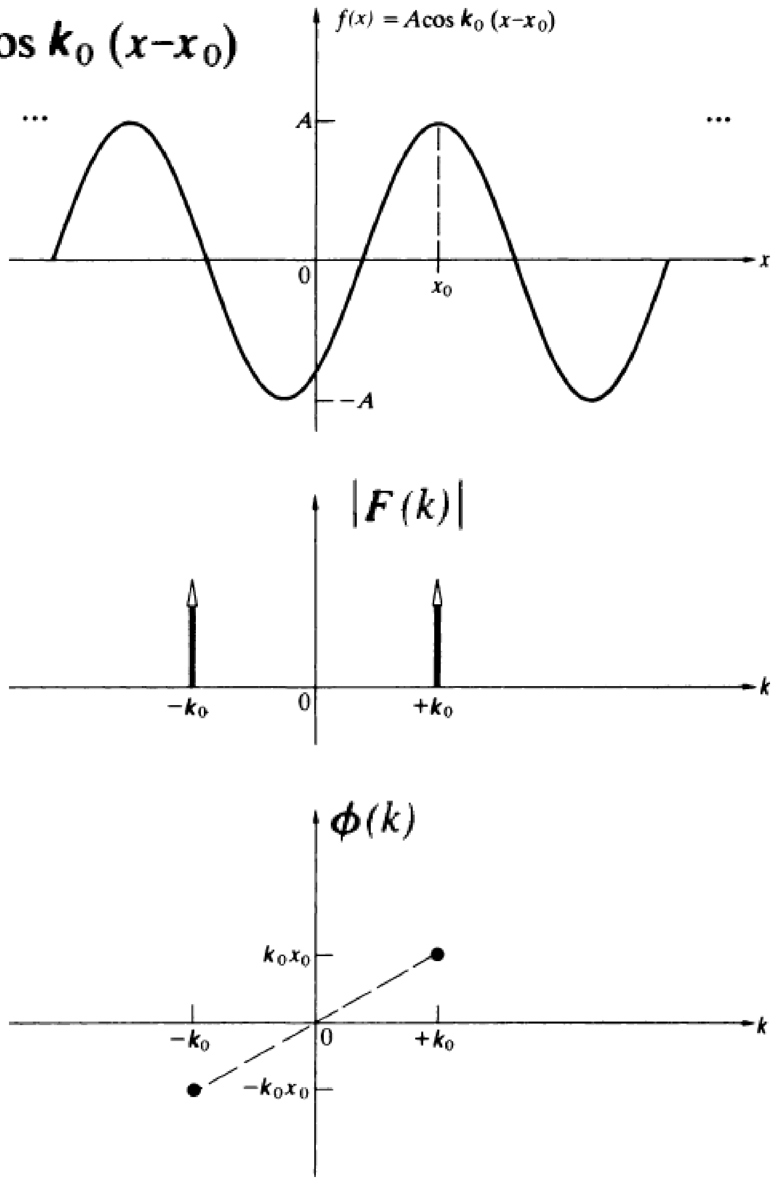
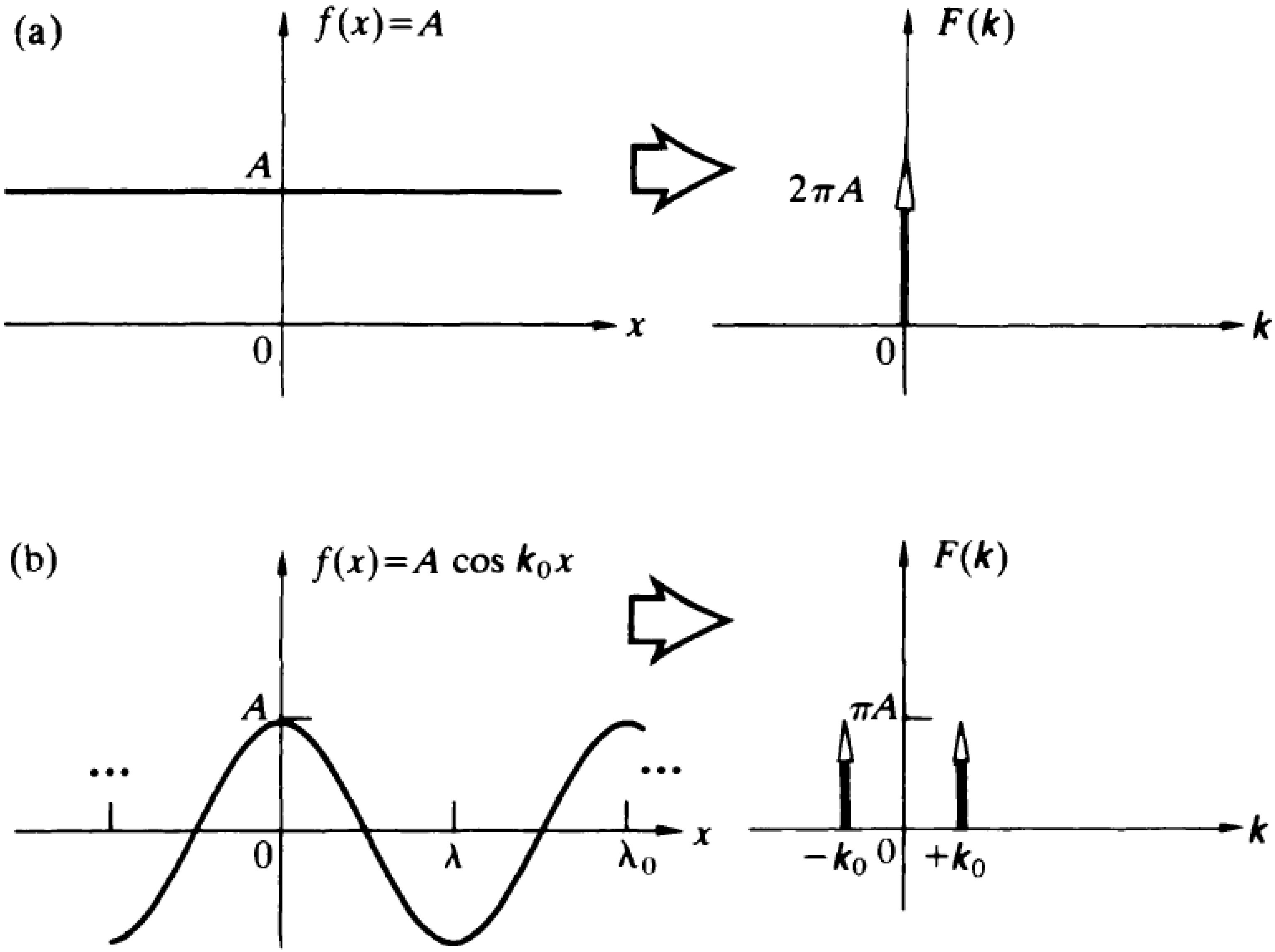


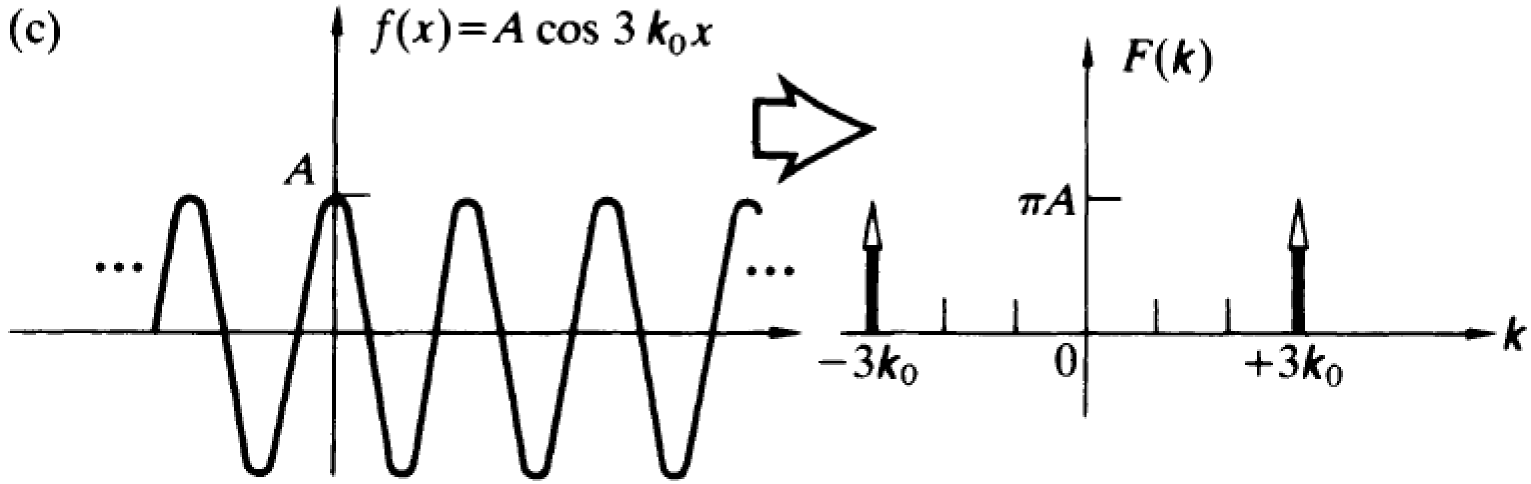
Figure 11.12 The spectra of a shifted cosine function

$$f(x) = A\cos k_0 (x-x_0)$$

Most functions, even harmonic ones, will usually be a blend of real and imaginary parts. For example, once a cosine is displaced a little, the new function, which is typically neither odd nor even, has both a real and an imaginary part. Moreover, it can be expressed as a cosinusoidal amplitude spectrum, which is appropriately phase-shifted (Fig. 11.12). Notice that when the cosine is shifted $\frac{1}{4}\lambda$ into a sine, the relative phase difference between the two component delta functions is again π rad.




(a)
f(x)=A
A
0
x
F(k)
2πA
0
k
(b)
f(x)=A cos k0x
A
...
...
0
λ
λ0
x
F(k)
πA
−k0
0
+k0
k


(c)
$f(x)=A\cos 3k_0x$
A
...
...
$F(k)$
πA
$-3k_0$
0
$+3k_0$
k

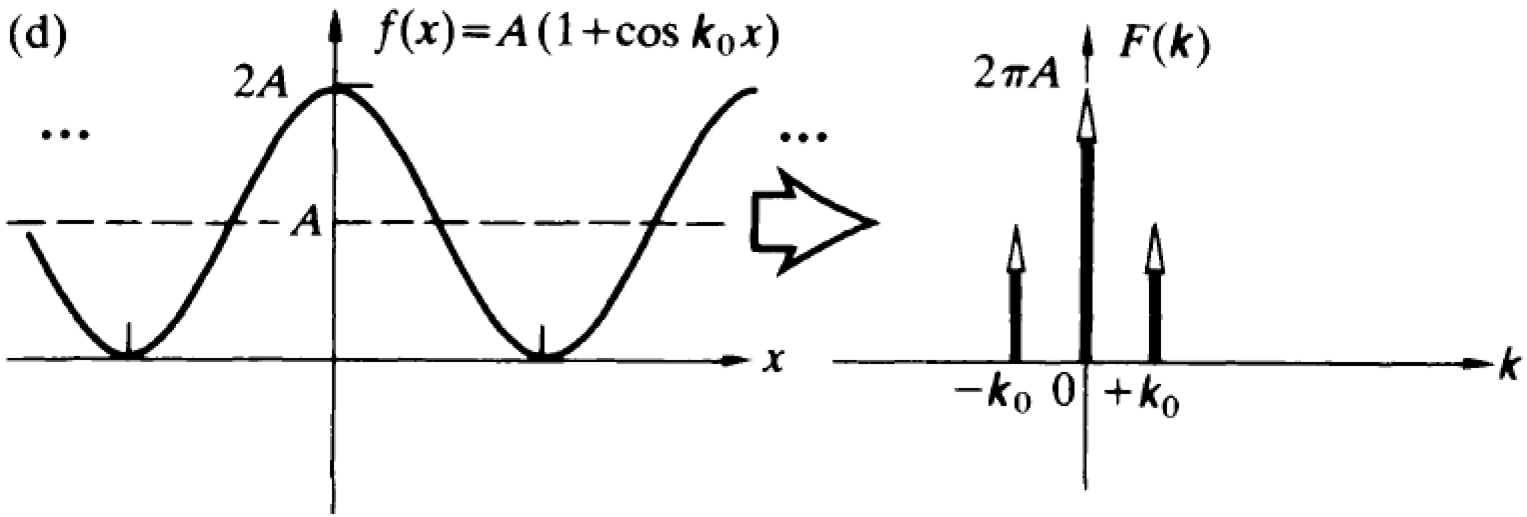

(d)
$f(x)=A(1+\cos k_0x)$
2A
...
...
A
x
$F(k)$
$2\pi A$
$-k_0$
0
$+k_0$
k

Figure 11.13 Some functions and their transforms.

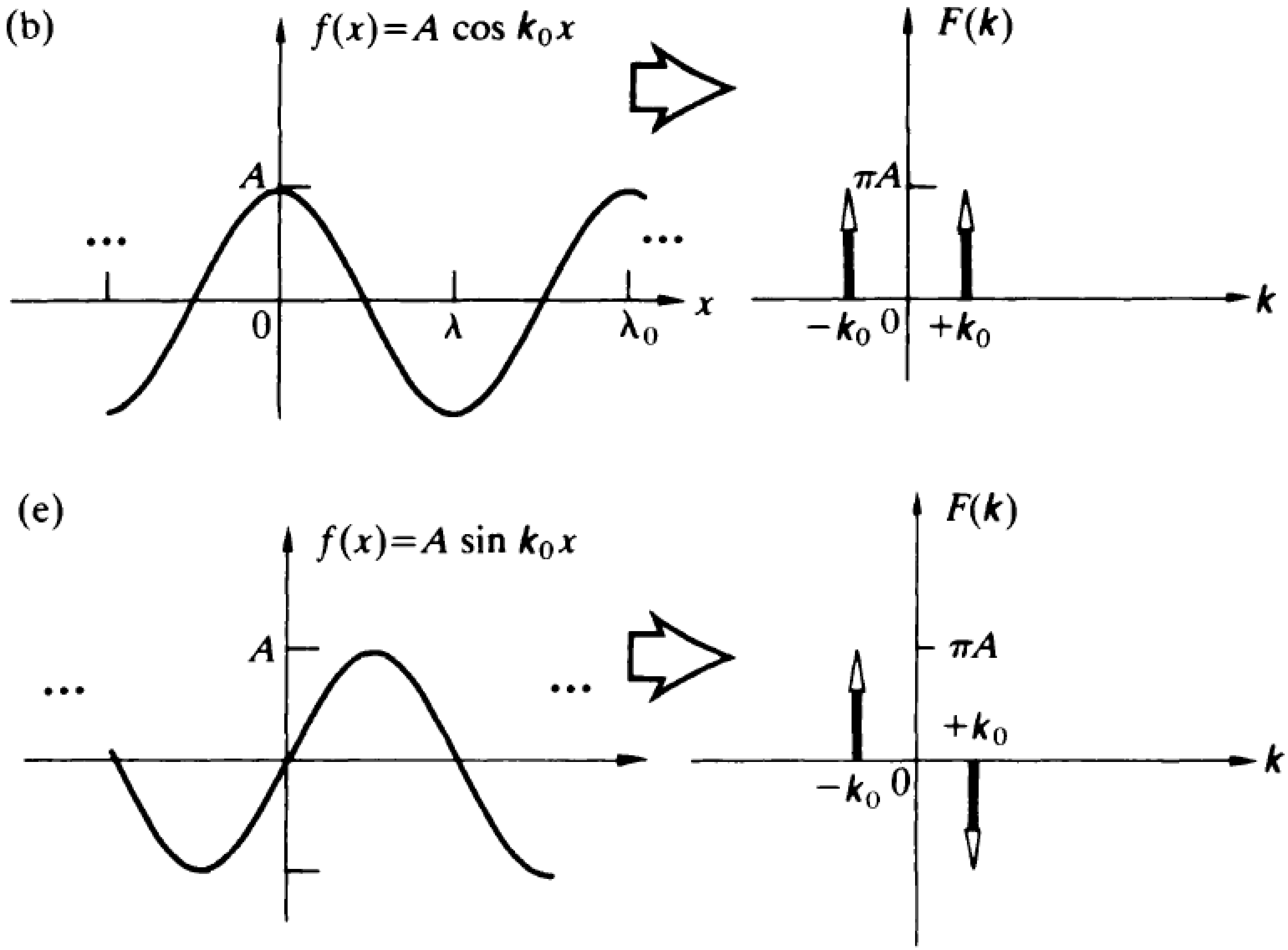
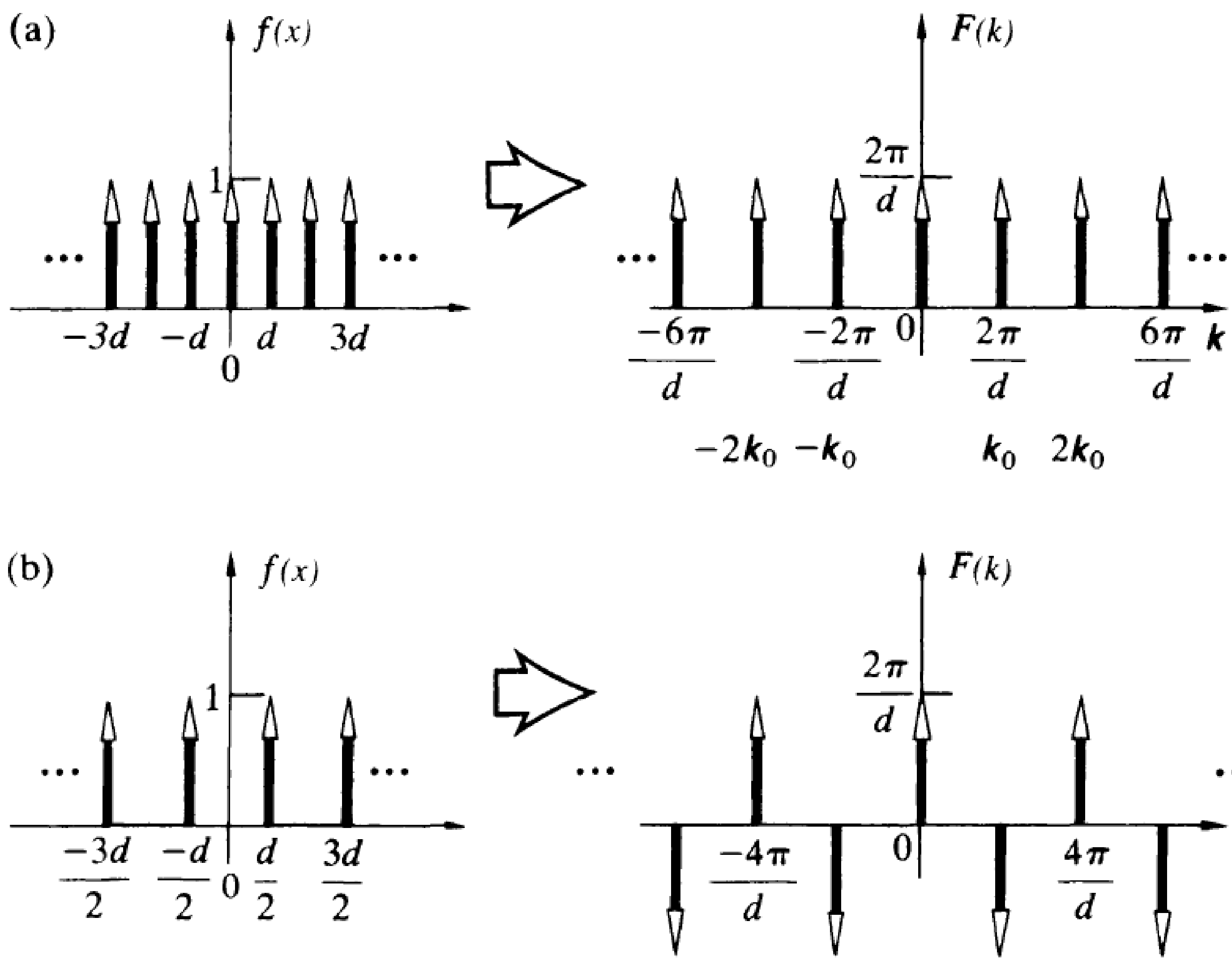


Figure 11.14 (*a*) The comb function and its transform
(*b*) A shifted comb function and its transform.



11.3 Optical Applications

11.3.1 Linear Systems

system is linear if:

1. multiplying $f(y, z)$ by a constant a produces an output $ag(Y, Z)$.
2. when the input is a weighted sum of two (or more) functions, $af_1(y, z) + bf_2(y, z)$, the output will similarly have the form $ag_1(Y, Z) + bg_2(Y, Z)$, where $f_1(y, z)$ and $f_2(y, z)$ generate $g_1(Y, Z)$ and $g_2(Y, Z)$ respectively.

Furthermore, a linear system will be *space invariant* if it possesses the property of *stationarity*; that is, in effect, changing the position of the input merely changes the location of the output without altering its functional form. The idea behind

impulse response.

the input and output can be written as

$$g(Y,Z) = \mathscr{L}\{f(y,z)\} \tag{11.46}$$

$$g(Y, Z) = \mathscr{L}\left\{\iint_{-\infty}^{+\infty} f(y', z')\delta(y' - y)\delta(z' - z)\, dy'\, dz'\right\}$$

$$g(Y, Z) = \iint_{-\infty}^{+\infty} f(y', z')\mathscr{L}\{\delta(y' - y)\delta(z' - z)\}\, dy'\, dz' \tag{11.47}$$

Consider the self-luminous and, therefore, incoherent source depicted in Fig. 11.15. We can imagine that each point on the object plane, Σ_0, emits light that is processed by the

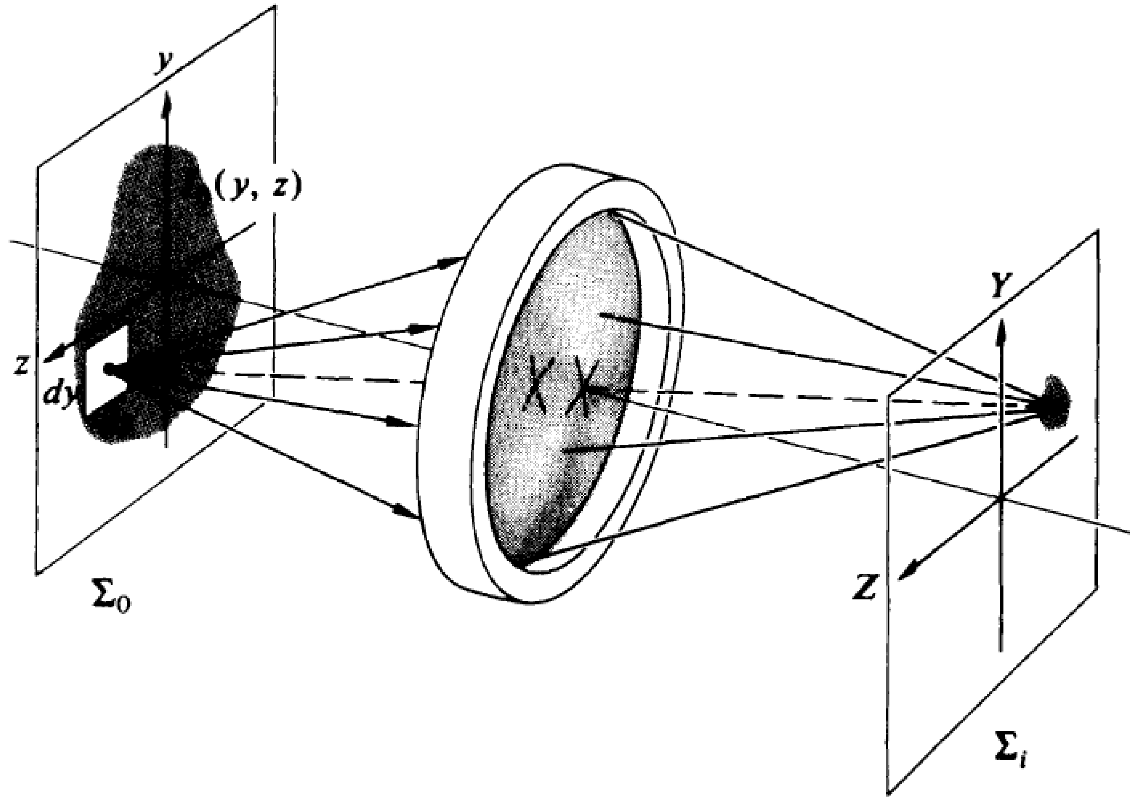


Figure 11.15 A lens system forming an image.

$$dI_i(Y, Z) = \mathcal{S}(y, z; Y, Z) I_0(y, z)\, dy\, dz \qquad (11.48)$$

$$I_i(Y, Z) = \iint_{-\infty}^{+\infty} I_0(y, z) \mathcal{S}(y, z; Y, Z)\, dy\, dz \qquad (11.49)$$

$$I_i(Y, Z) = A \iint_{-\infty}^{+\infty} \delta(y - y_0) \delta(z - z_0) \mathcal{S}(y, z; Y, Z)\, dy\, dz$$

from the sifting property $\Rightarrow$ $\boxed{I_i(Y, Z) = A \mathcal{S}(y_0, z_0; Y, Z)}$

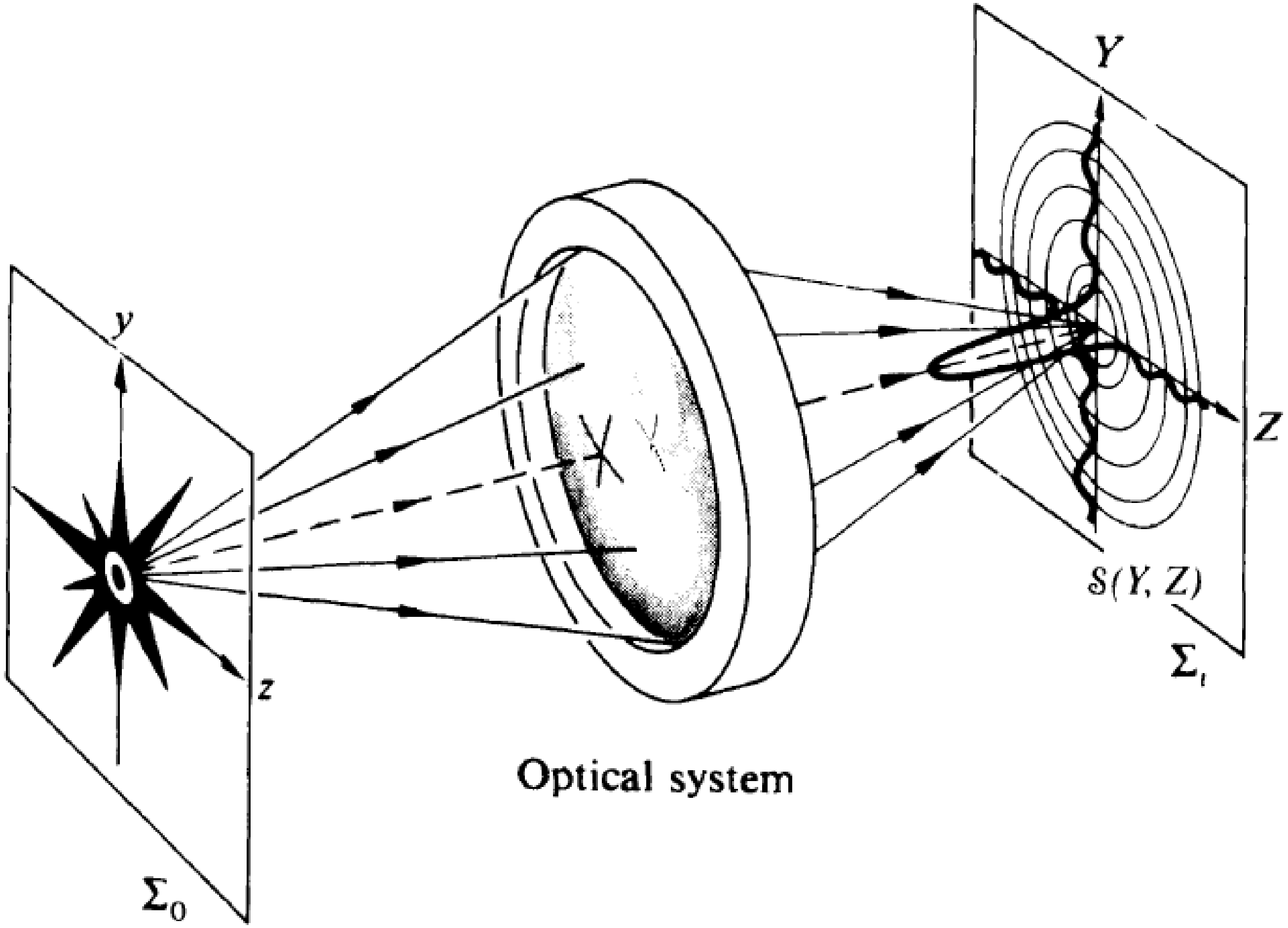
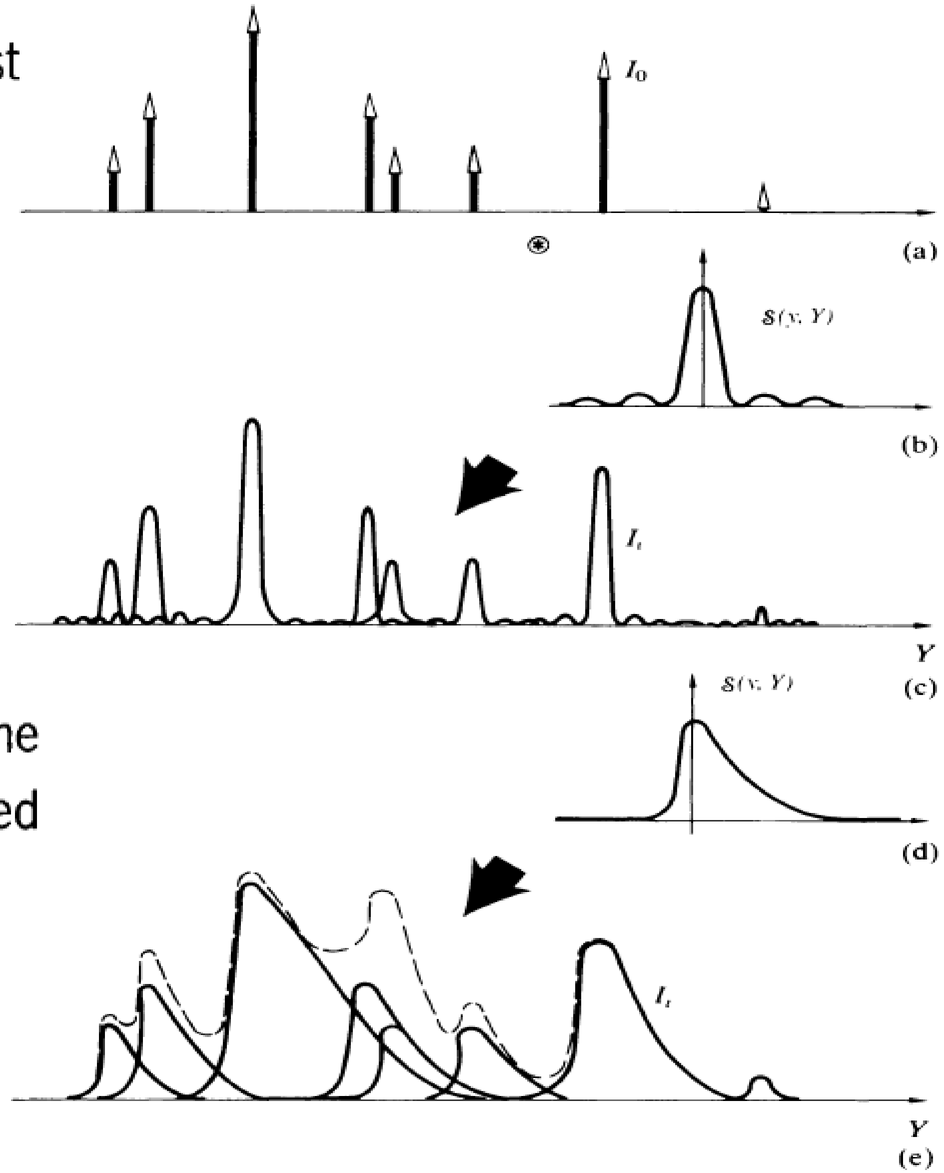


Figure 11.16 The point-spread function: the irradiance produced by the optical system with an input point source.

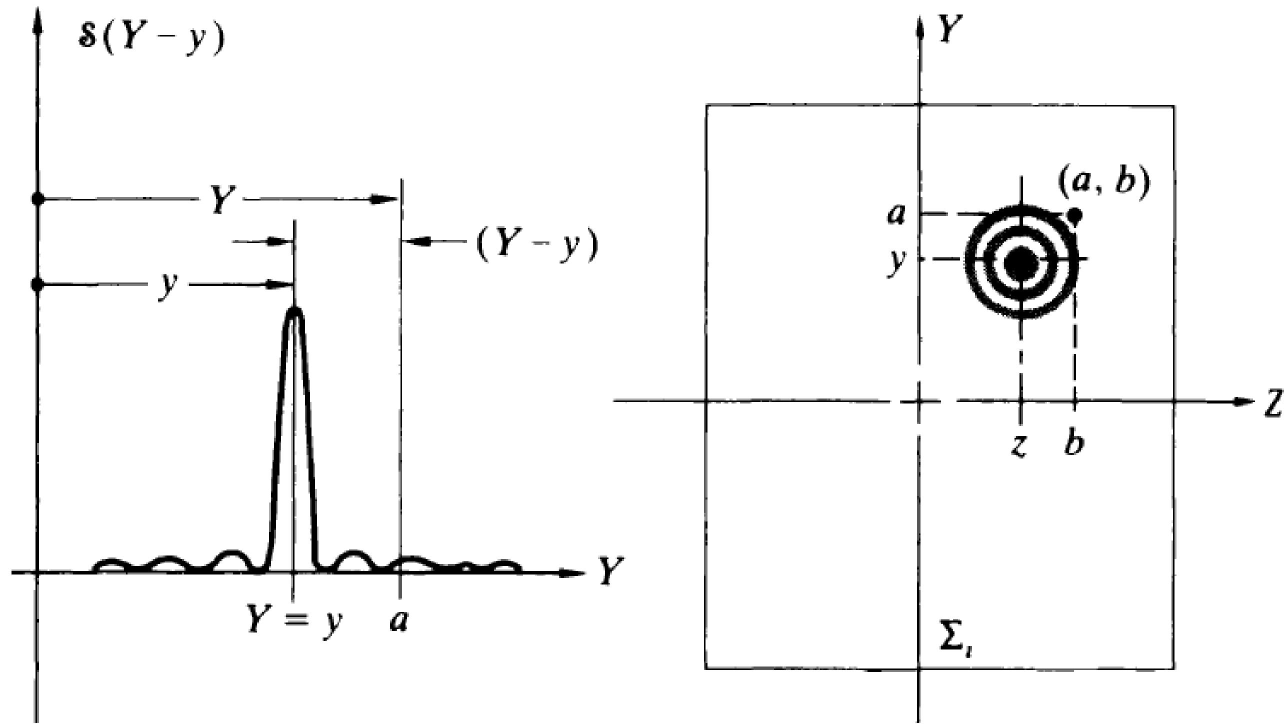
Figure 11.17 Here (a) is convolved first with (b) to produce (c) and then with (d) to produce (e).



The resulting pattern is the sum of all the spread-out contributions as indicated by the dashed curve in (e).

Figure 11.18

The point-spread function.



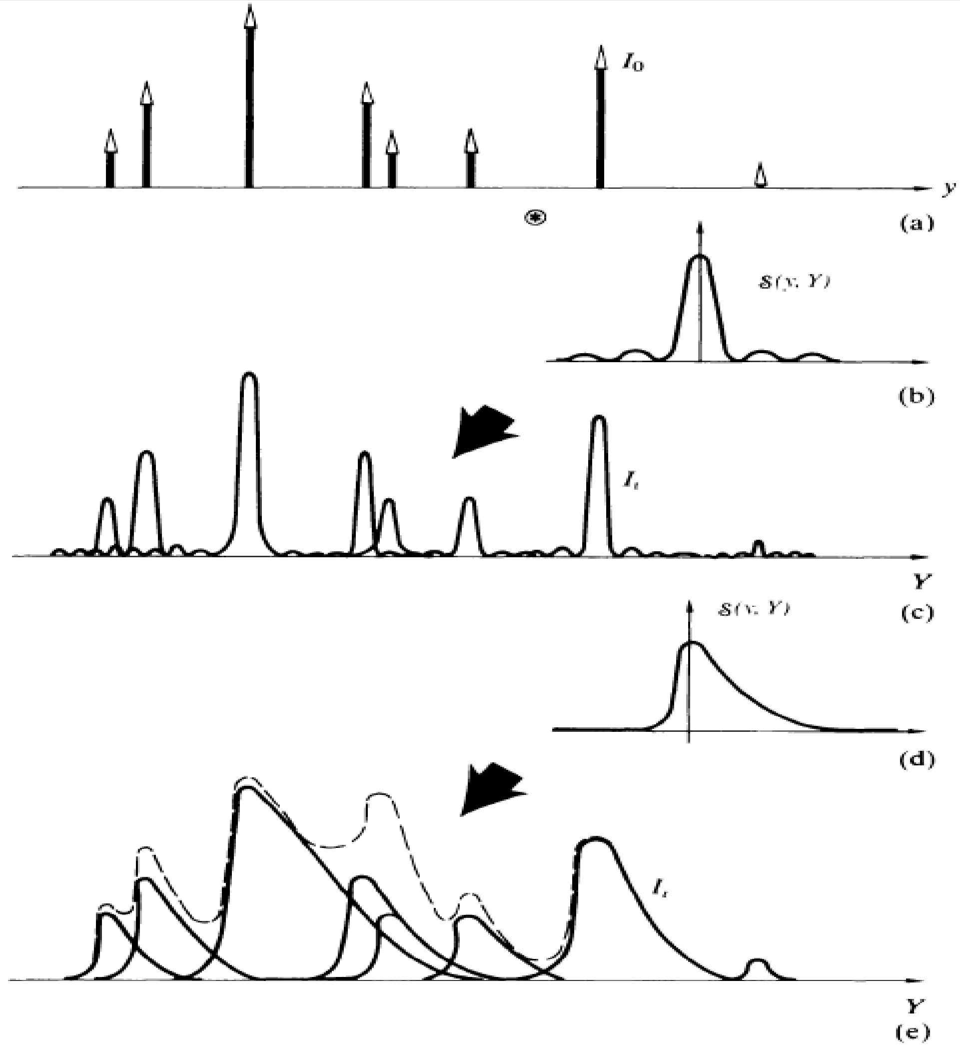
$$\mathcal{S}(y, z; Y, Z) = \mathcal{S}(Y-y, Z-z) \quad (11.50)$$

$$I_i(Y, Z) = \iint_{-\infty}^{+\infty} I_0(y, z)\mathcal{S}(Y-y, Z-z)\, dy\, dz \quad (11.51)$$

11.3.2 The Convolution Integral

$\boxed{\text{one-dimensional}}$

$$g(X) = \int_{-\infty}^{+\infty} f(x)h(X - x)\, dx \tag{11.52}$$



11.3.2 The Convolution Integral

two dimensions

$$I_i(Y,Z) = \int\int_{-\infty}^{+\infty} I_0(y,z)\mathcal{S}(Y-y, Z-z)\, dy\, dz \qquad (11.51)$$

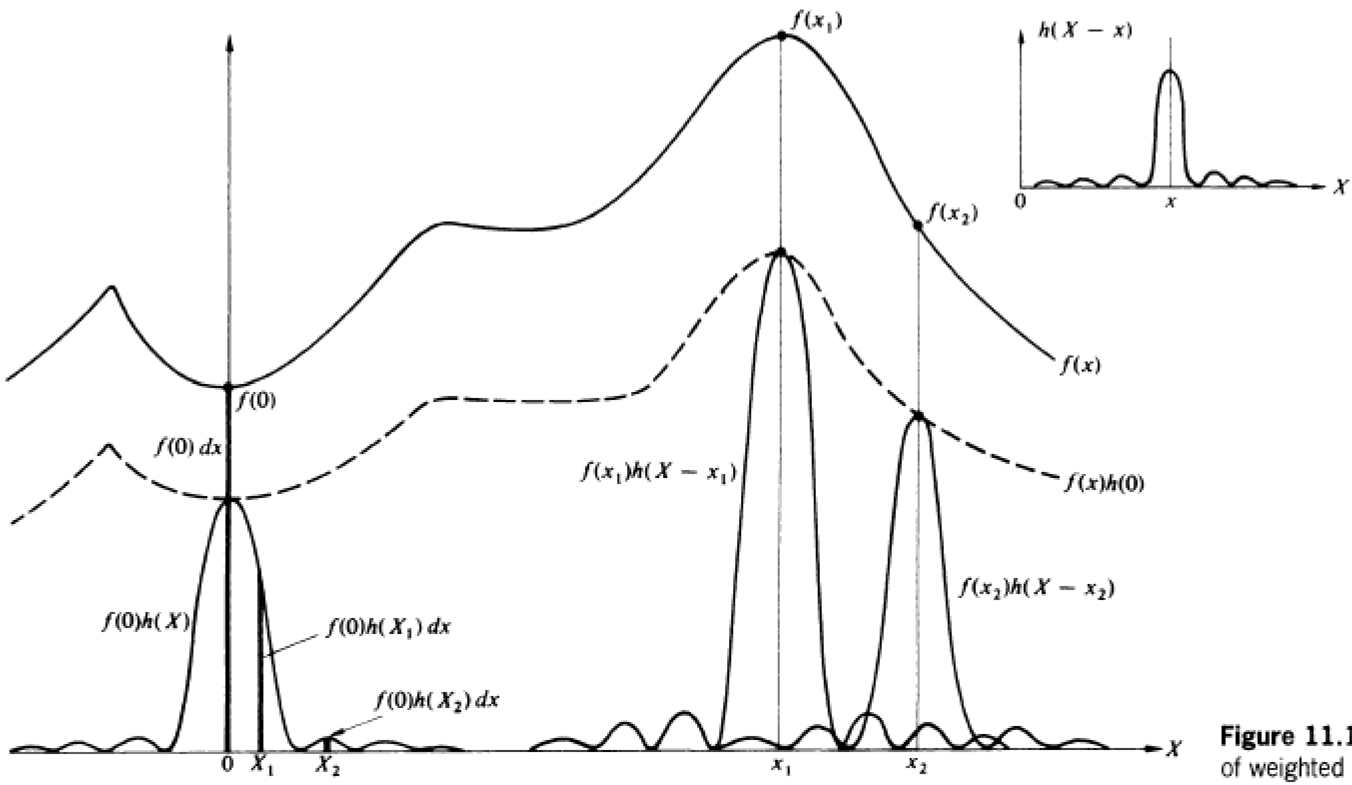


Figure 11.19 The overlapping of weighted spread functions.

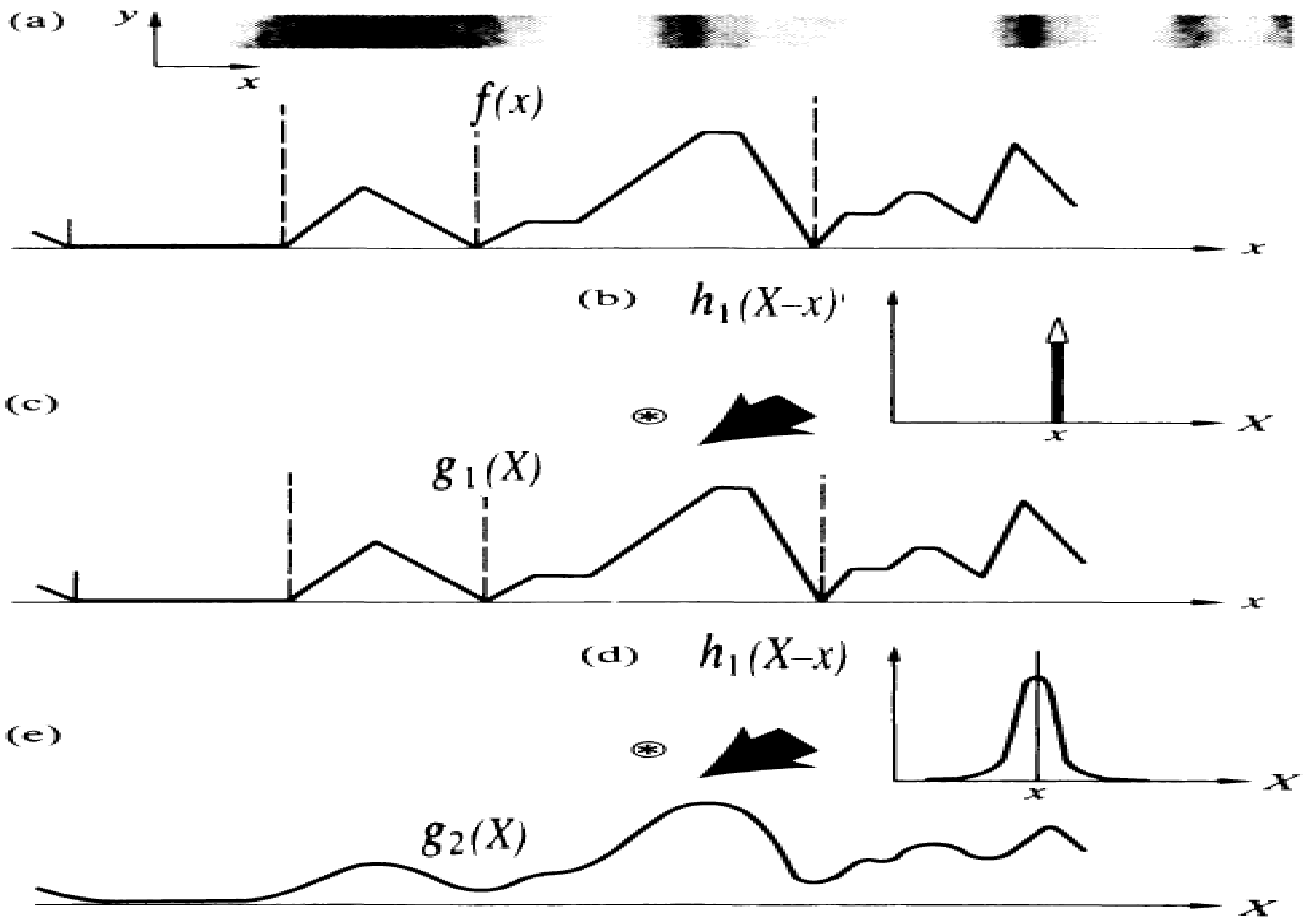


Figure 11.20 The irradiance distribution is converted to a function $f(x)$ shown in (a). This is convolved with a δ-function (b) to yield a duplicate of $f(x)$. By contrast, convolving $f(x)$ with the spread function h_2 in (d) yields a smoothed-out curve represented by $g_2(x)$ in (e).

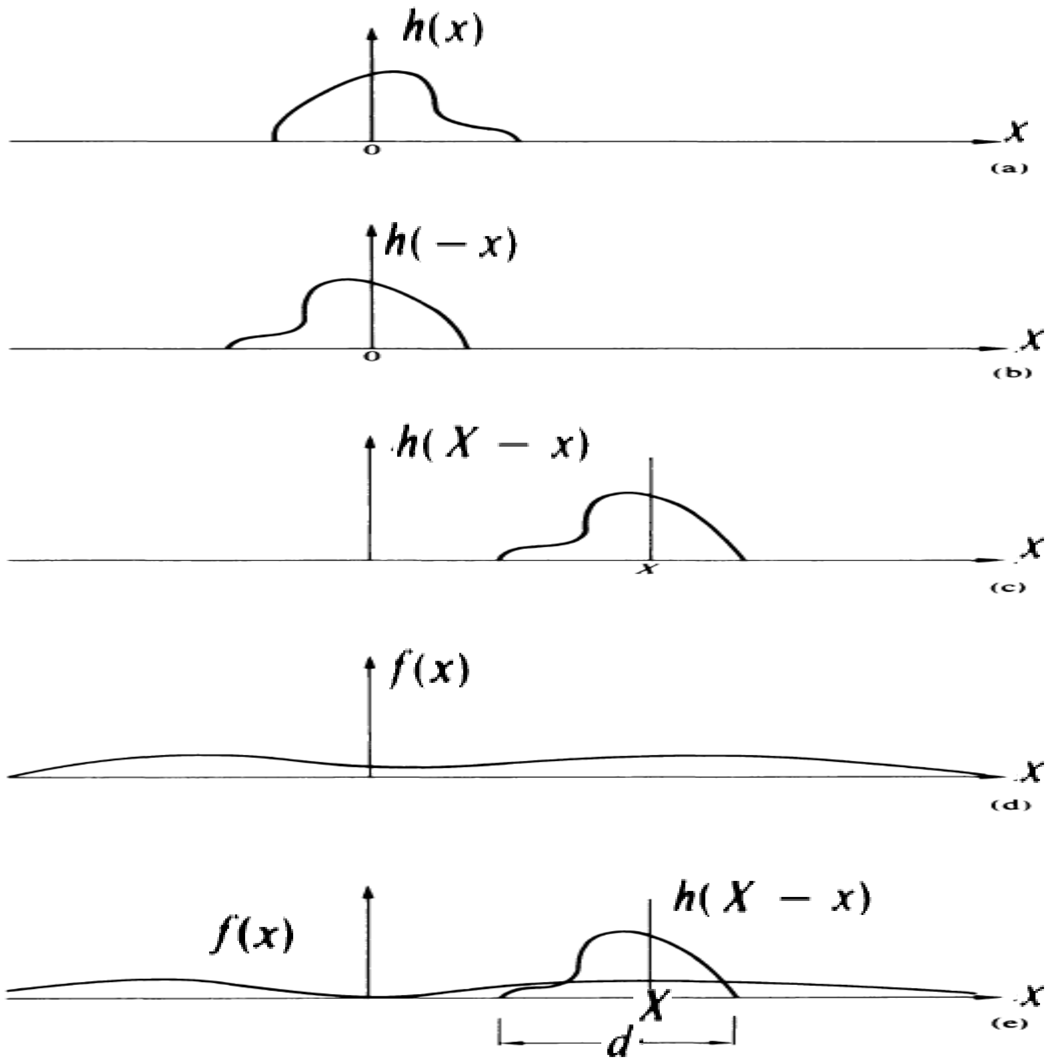


Figure 11.21 The geometry of the convolution process in the object coordinates.

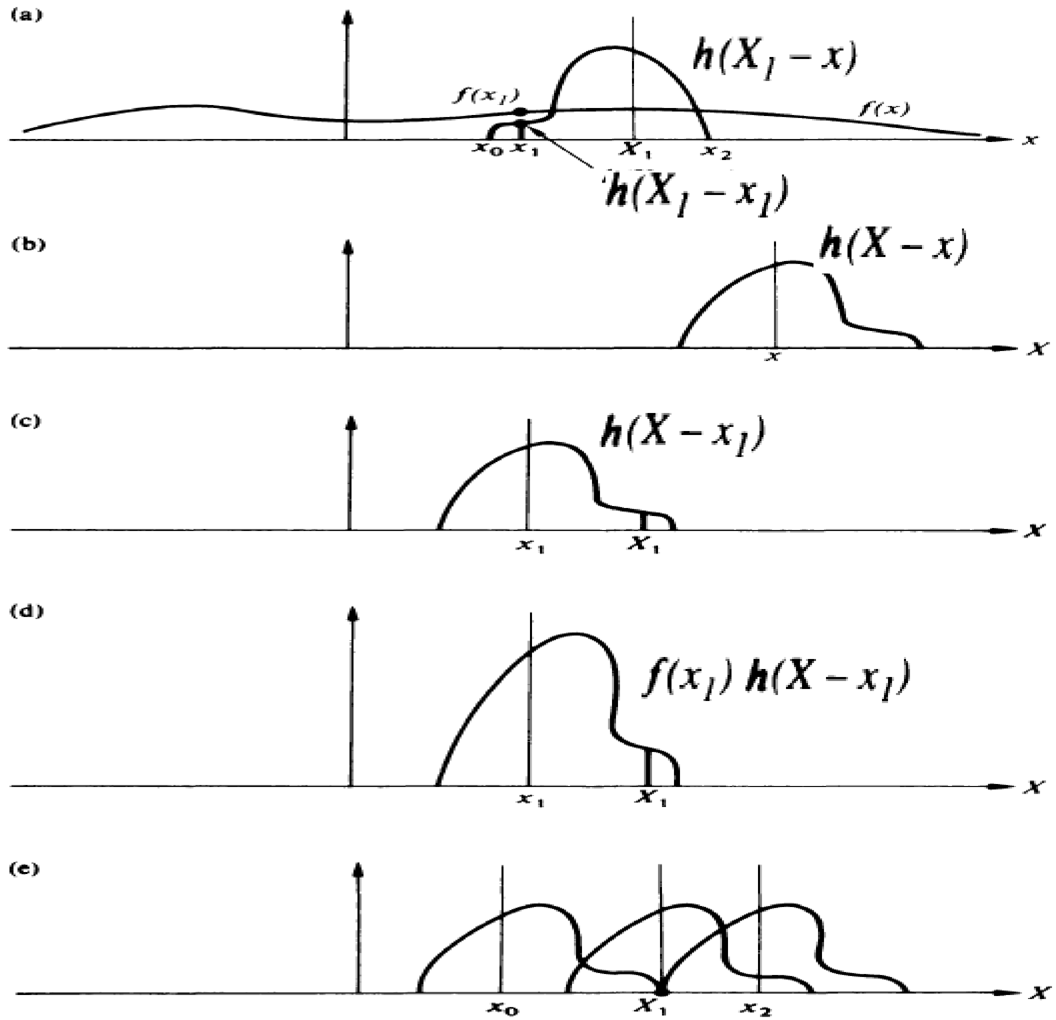


Figure 11.22 The geometry of the convolution process in the image coordinates.

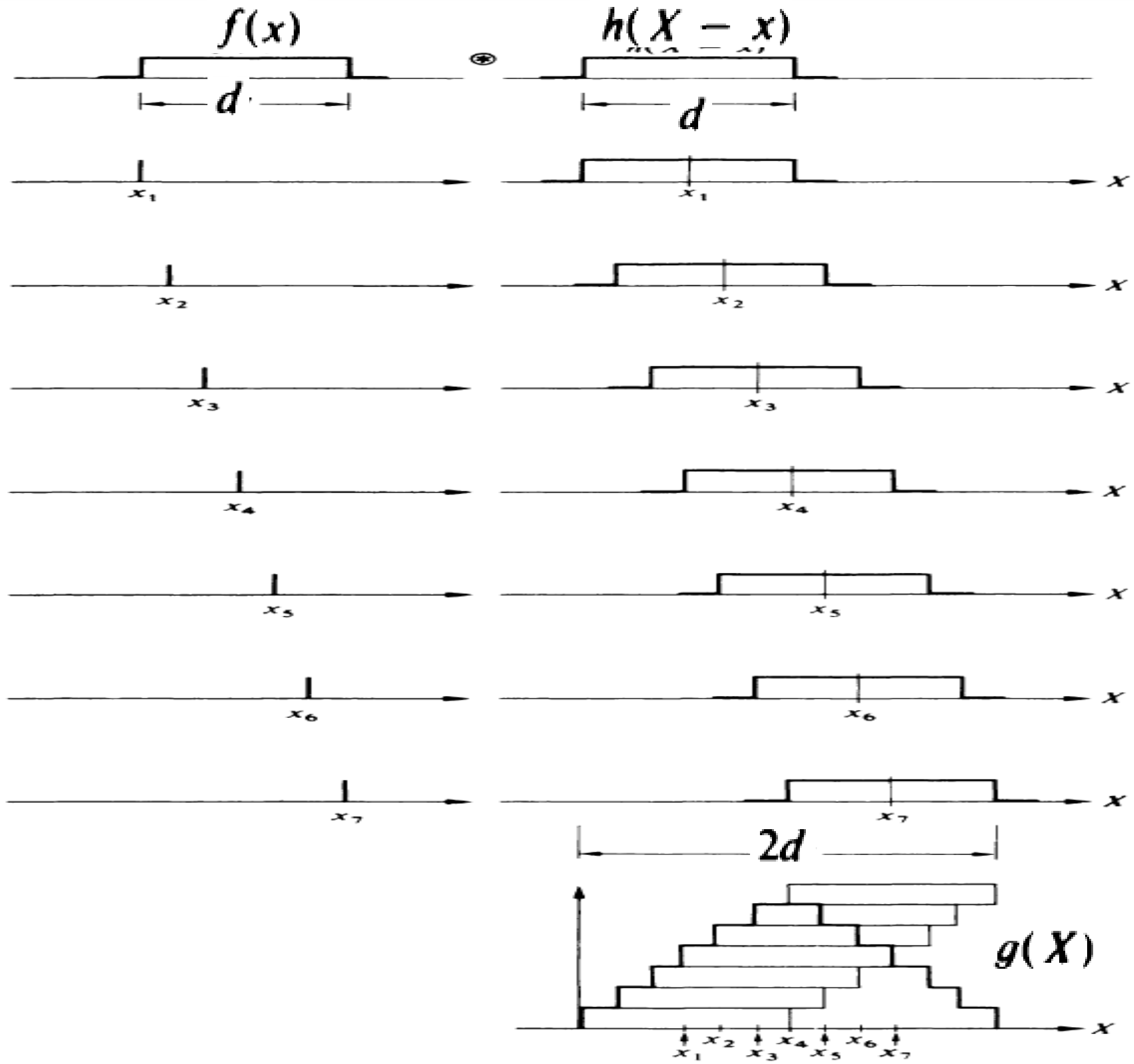


Figure 11.23 Convolution of two square pulses. The fact that we represented $f(x)$ by a finite number of delta functions (viz., 7) accounts for the steps in $g(X)$.

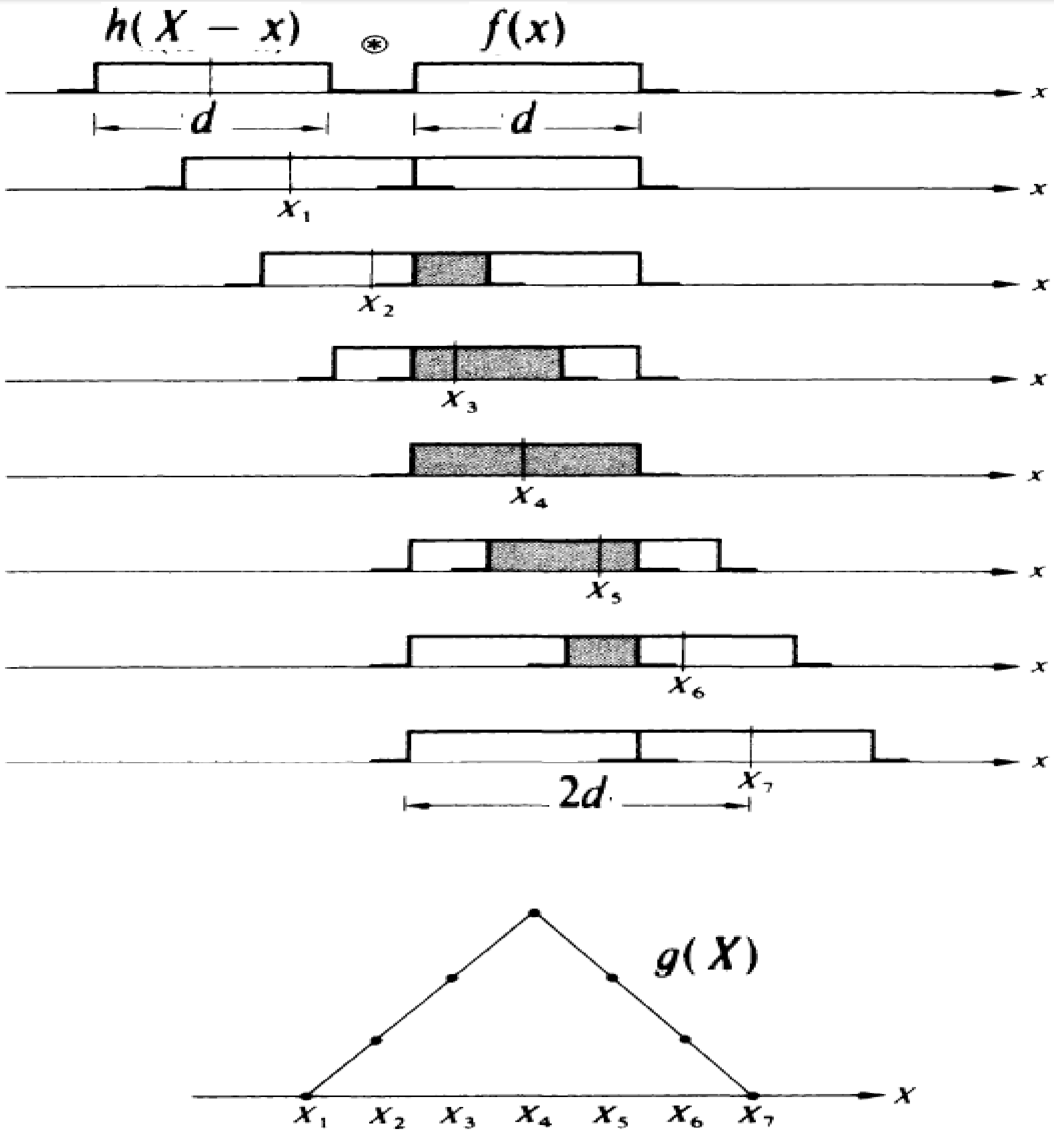


Figure 11.24 Convolution of two square pulses.

Here the volume under the product curve $I_0(y, z)\mathcal{S}(Y - y, Z - z)$, that is, the region of overlap, equals $I_i(Y, Z)$ at (Y, Z);

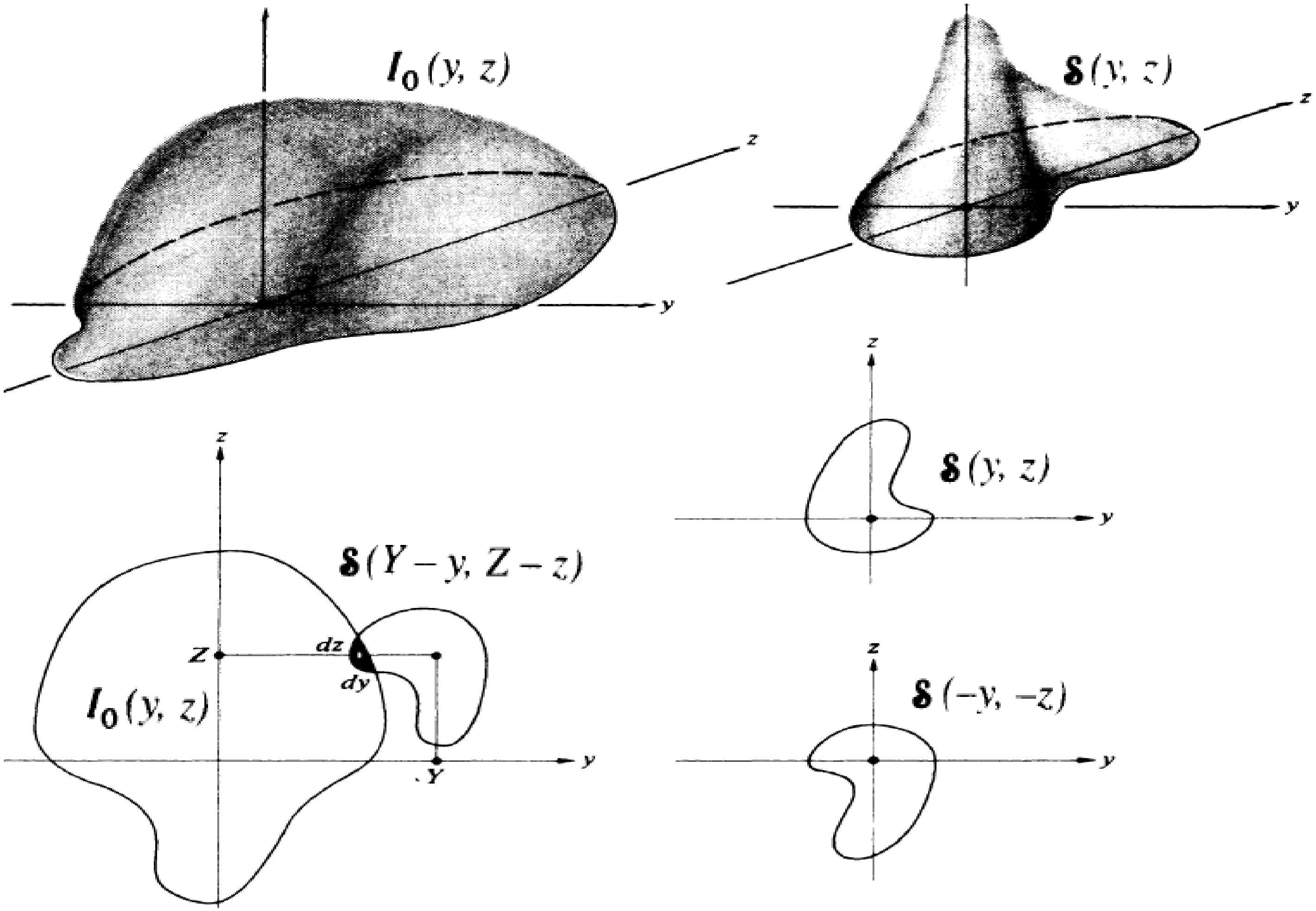


Figure 11.25 Convolution in two dimensions.

The Convolution Theorem

Suppose we have two functions $f(x)$ and $h(x)$ with Fourier transforms $\mathscr{F}\{f(x)\} = F(k)$ and $\mathscr{F}\{h(x)\} = H(k)$, respectively. The **convolution theorem** states that if $g = f \circledast h$,

$$\mathscr{F}\{g\} = \mathscr{F}\{f \circledast h\} = \mathscr{F}\{f\} \cdot \mathscr{F}\{h\} \qquad (11.53)$$

or

$$G(k) = F(k)H(k) \qquad (11.54)$$

where $\mathscr{F}\{g\} = G(k)$. The proof is quite straightforward:

$$\mathscr{F}\{f \circledast h\} = \int_{-\infty}^{+\infty} g(X)e^{ikX}\,dX$$

$$= \int_{-\infty}^{+\infty} e^{ikX}\left[\int_{-\infty}^{+\infty} f(x)h(X-x)\,dx\right]dX$$

$$\boxed{\begin{aligned}\mathscr{F}\{f \circledast h\} &= \int_{-\infty}^{+\infty} g(X) e^{ikX} dX \\ &= \int_{-\infty}^{+\infty} e^{ikX} \left[\int_{-\infty}^{+\infty} f(x) h(X - x) dx \right] dX\end{aligned}}$$

Thus

$$G(k) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} h(X - x) e^{ikX} dX \right] f(x)\, dx$$

If we put $w = X - x$ in the inner integral, then $dX = dw$ and

$$G(k) = \int_{-\infty}^{+\infty} f(x) e^{ikx} dx \int_{-\infty}^{+\infty} h(w) e^{ikw} dw$$

Hence

$$G(k) = F(k) H(k)$$

$$\mathcal{F}\{g\} = [d \text{ sinc } (kd/2)]^2 \tag{11.55}$$

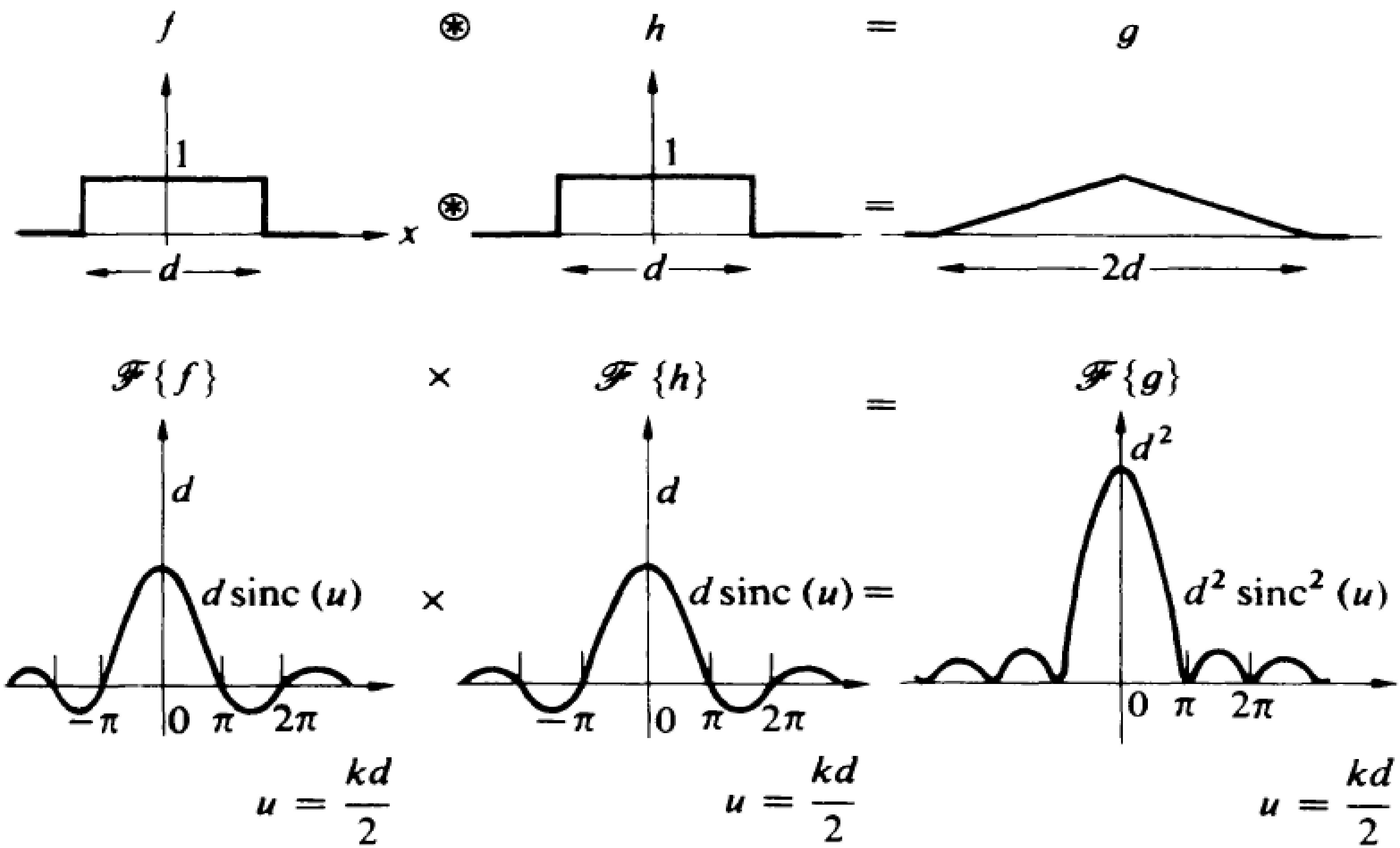


Figure 11.26 An illustration of the convolution theorem.

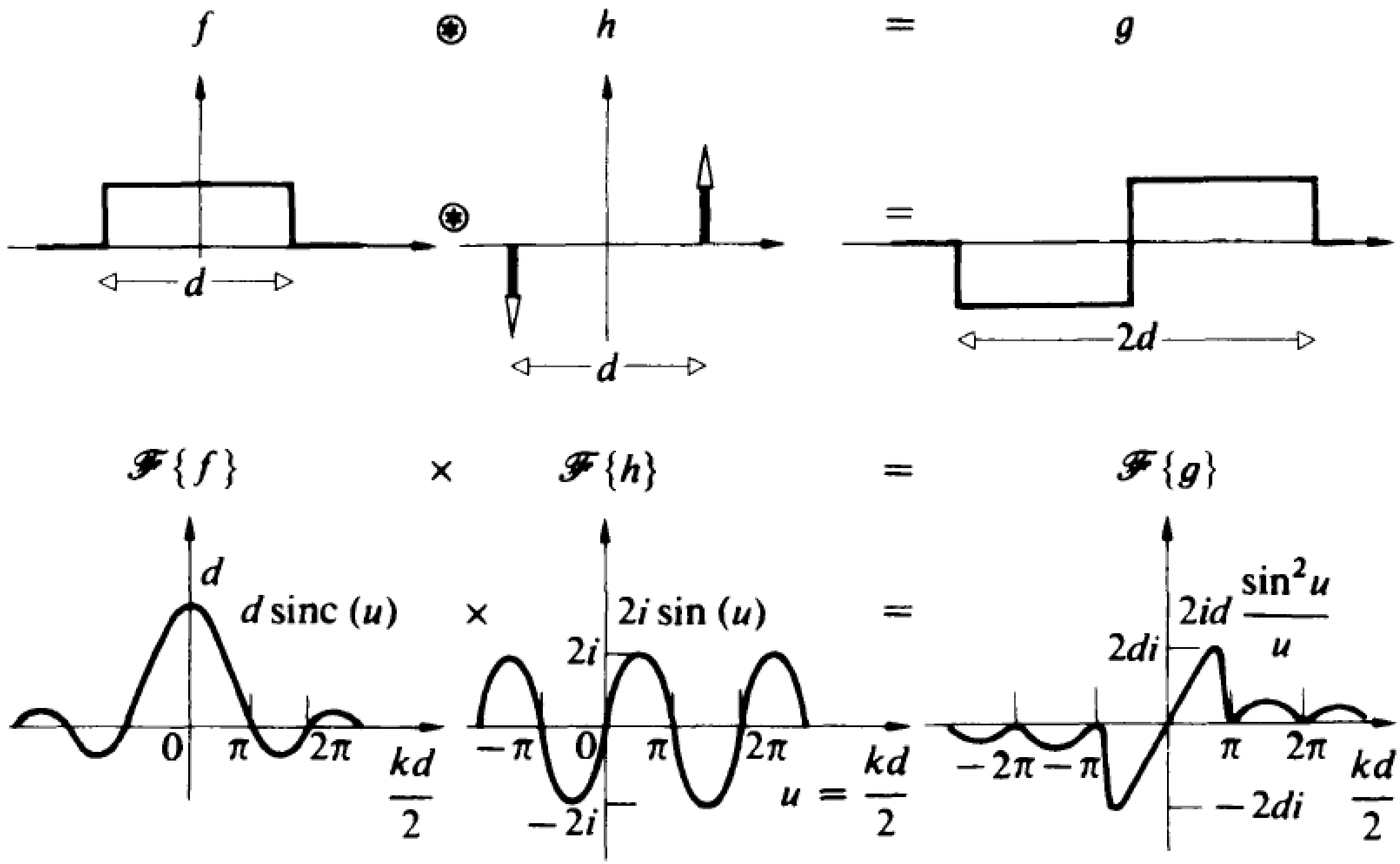


Figure 11.27 An illustration of the convolution theorem.

Frequency Convolution Theorem

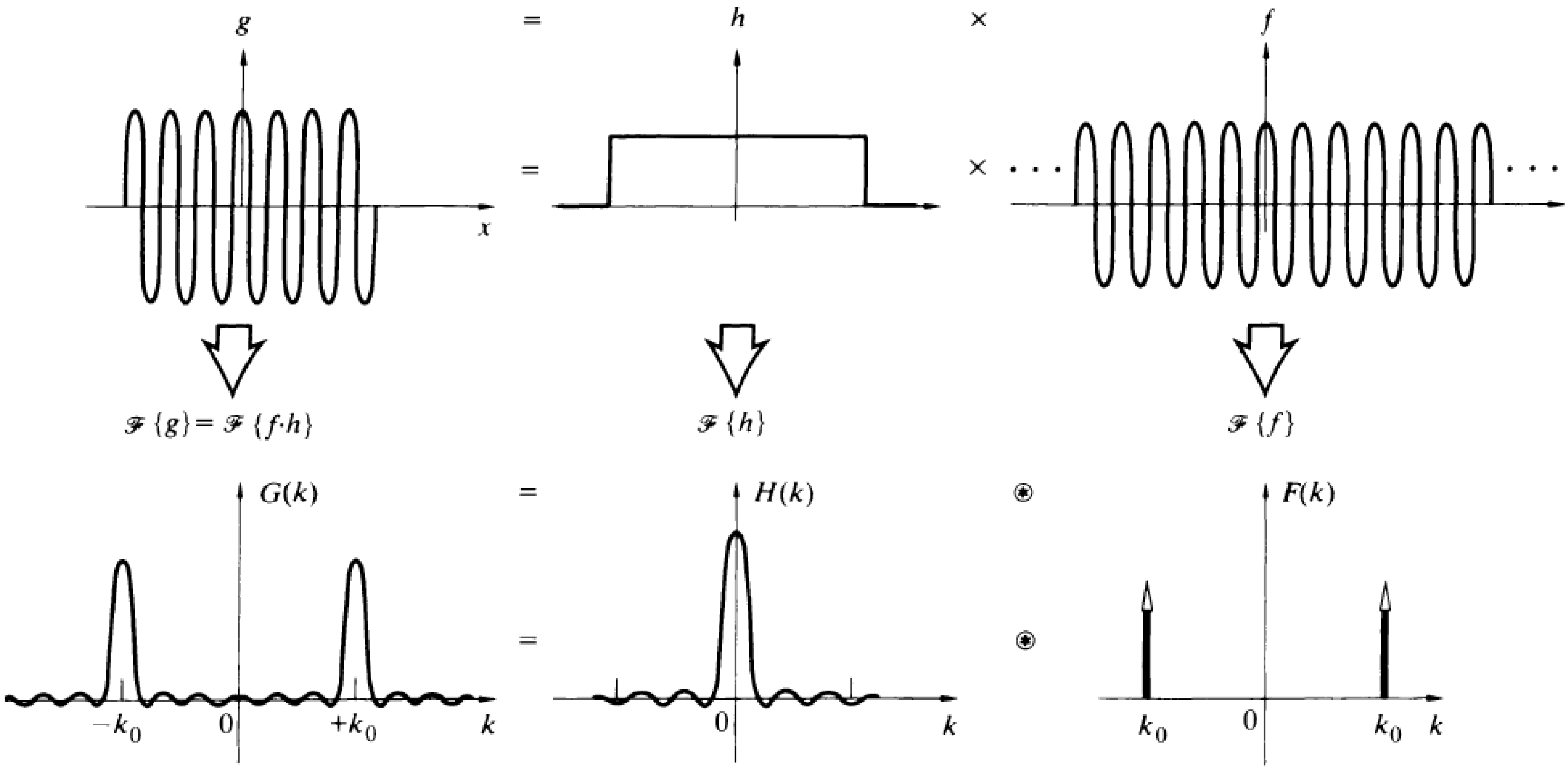


Figure 11.28
An example of the frequency convolution theorem.

$$\mathcal{F}\{f \cdot h\} = \frac{1}{2\pi}\mathcal{F}\{f\} \circledast \mathcal{F}\{h\} \qquad (11.56)$$

Transform of the Gaussian Wave Packet

$$\boxed{\tilde{E}(x,t) = E_0 e^{-i(k_0 x - \omega t)}}$$

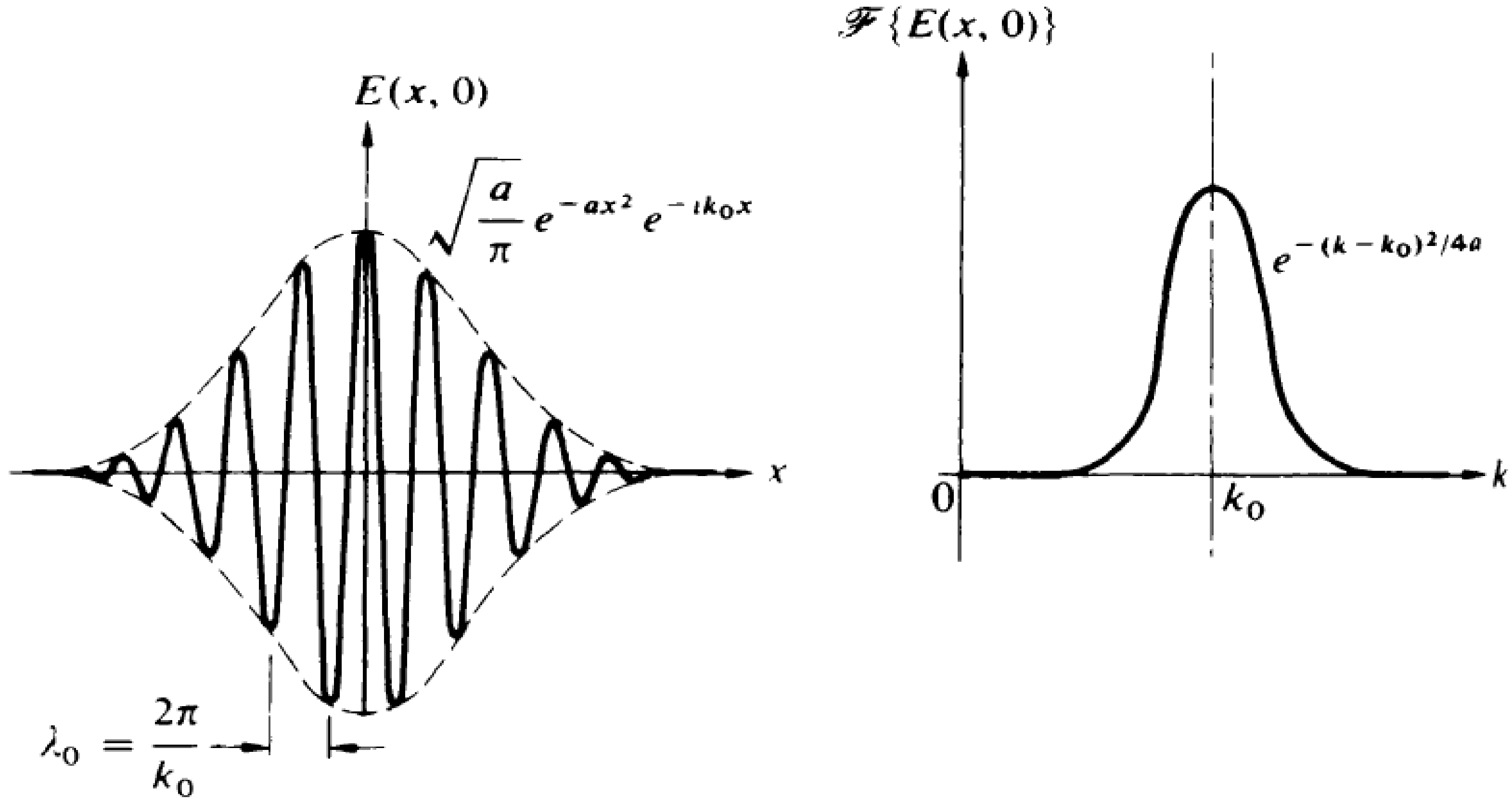


Figure 11.29 A Gaussian wave packet and its transform.

$$\boxed{\tilde{E}(x,t) = E_0 e^{-i(k_0 x - \omega t)}}$$

dent of time, we can write it as

$$\tilde{E}(x,0) = f(x) e^{-ik_0 x}$$

Now, to determine $\mathscr{F}\{f(x)e^{-ik_0 x}\}$ evaluate

$$\int_{-\infty}^{+\infty} f(x) e^{-ik_0 x} e^{ikx}\, dx \tag{11.57}$$

Letting $k' = k - k_0$, we get

$$F(k') = \int_{-\infty}^{+\infty} f(x) e^{ik'x}\, dx = F(k - k_0) \tag{11.58}$$

In other words, if $F(k) = \mathscr{F}\{f(x)\}$, then $F(k - k_0) = \mathscr{F}\{f(x)e^{-ik_0x}\}$. For the specific case of a Gaussian envelope [Eq. (11.11)], as in the figure, $f(x) = \sqrt{a/\pi}\, e^{-ax^2}$, that is,

$$\tilde{E}(x,0) = \sqrt{a/\pi}\, e^{-ax^2} e^{-ik_0x} \tag{11.59}$$

From the foregoing discussion and Eq. (11.12), it follows that

$$\mathscr{F}\{\tilde{E}(x,0)\} = e^{-(k-k_0)^2/4a} \tag{11.60}$$

11.3.3 Fourier Methods in Diffraction Theory

Fraunhofer Diffraction

$$E(Y,Z) = \frac{\mathcal{E}_A e^{i(\omega t - kR)}}{R} \iint_{\text{Aperture}} e^{ik(Yy+Zz)/R}\, dy\, dz \qquad (11.61)$$

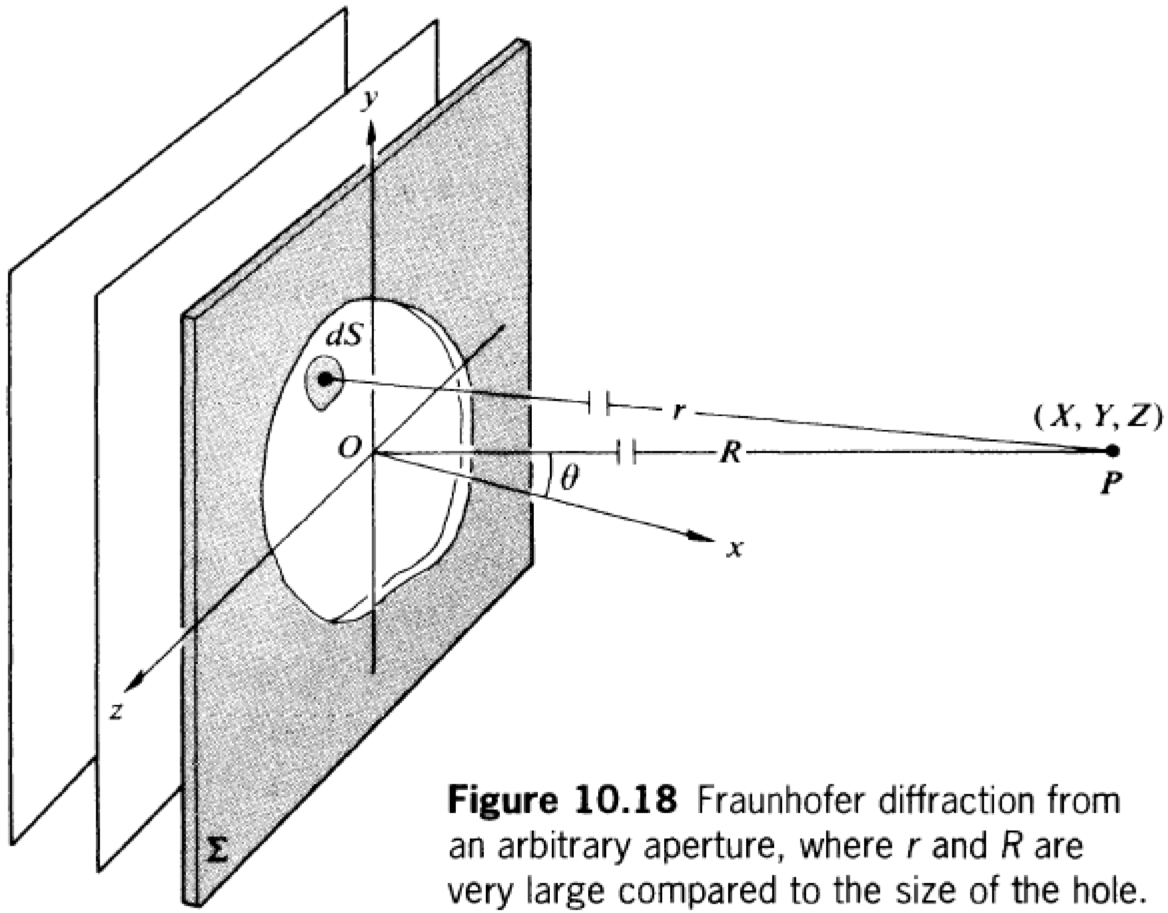


Figure 10.18 Fraunhofer diffraction from an arbitrary aperture, where r and R are very large compared to the size of the hole.

aperture function
$$\mathcal{A}(y,z) = \mathcal{A}_0(y,z)e^{i\phi(y,z)} \tag{11.62}$$

$$E(Y,Z) = \iint\limits_{-\infty}^{+\infty} \mathcal{A}(y,z)e^{ik(Yy+Zz)/R}\,dy\,dz \tag{11.63}$$

and k_Z *as*

$$k_Y \equiv kY/R = k\sin\phi = k\cos\beta \tag{11.64}$$

and
$$k_Z \equiv kZ/R = k\sin\theta = k\cos\gamma \tag{11.65}$$

For each point on the image plane, there is a corresponding spatial frequency. The diffracted field can now be written as

$$E(k_Y,k_Z) = \iint\limits_{-\infty}^{+\infty} \mathcal{A}(y,z)e^{i(k_Y y+k_Z z)}\,dy\,dz \tag{11.66}$$

and we've arrived at the key point: ***the field distribution in the Fraunhofer diffraction pattern is the Fourier transform of the field distribution across the aperture*** (***i.e., the aperture function***). Symbolically, this is written as

$$E(k_Y, k_Z) = \mathscr{F}\{\mathscr{A}(y,z)\} \tag{11.67}$$

The field distribution in the image plane is the spatial-frequency spectrum of the aperture function. The inverse transform is then

$$\mathscr{A}(y,z) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} E(k_Y, k_Z) e^{-i(k_Y y + k_Z z)} \, dk_Y \, dk_Z \tag{11.68}$$

that is,

$$\mathscr{A}(y,z) = \mathscr{F}^{-1}\{E(k_Y, k_Z)\} \tag{11.69}$$