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The weighted Davis-Wielandt Berezin number for reproducing kernel Hilbert space operators

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Abstract

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathcal{C}$ that the evaluation functionals $\varphi_\lambda(f) = f(\lambda)$, $\lambda \in \Theta$ are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem, there is a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$ and every $\lambda \in \Theta$. The function k on $\Theta \times \Theta$ defined by $k(z, \lambda) = k_\lambda(z)$ is called the reproducing kernel of $\mathcal{H}$. In this study, we defined the weighted Davis-Wielandt Berezin number, and then we obtained some related inequalities. It is shown, among other inequalities, that if $X \in \mathcal{L}(\mathcal{H})$ and $\nu \in [0, 1]$, then

$$\begin{aligned} & \frac{1}{2} \left(\text{ber}^2(X_\nu + |X_\nu|^2) + c_{\text{ber}}^2(X_\nu - |X_\nu|^2) \right) \\ & \leq dw_{\text{ber}, \nu}^2(X) \\ & \leq \frac{1}{2} \left(\text{ber}^2(X_\nu + |X_\nu|^2) + \text{ber}^2(X_\nu - |X_\nu|^2) \right), \end{aligned}$$

where $X_\nu = (1 - 2\nu)X^* + X$. Some bounds for the weighted Davis-Wielandt Berezin number are also established.

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1 Introduction and preliminaries

Let $\mathcal{L}(\mathcal{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ with the identity operator $I_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{H})$. Recall that the numerical range and the numerical radius of $X \in \mathcal{L}(\mathcal{H})$ are defined by $W(X) := \{ \langle Xy, y \rangle : y \in \mathcal{H} \text{ and } \|y\| = 1 \}$ and $w(X) := \sup \{ |\tau| : \tau \in W(X) \}$, respectively.

For further details on the numerical radius, see [5, 23–25] and references therein. Moreover, one of the most less commonly celebrated generalizations of the numerical range and the numerical radius is the Davis-Wielandt shell and its radius of $X \in \mathcal{L}(\mathcal{H})$, which are de-

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defined as:

$$DW(X) := \{(\langle Xy, y \rangle, \langle Xy, Xy \rangle) : y \in \mathcal{H} \text{ \& \; } \|y\| = 1\},$$

and

$$\begin{aligned} dw(X) &:= \sup \{|\tau| : \tau \in DW(X)\} \\ &= \sup \left\{ \sqrt{|\langle Xy, y \rangle|^2 + \|Xy\|^4} : y \in \mathcal{H} \text{ \& \; } \|y\| = 1 \right\}. \end{aligned} \quad (1)$$

The following inequality immediately comes from (1):

$$\max\{w(X), \|X\|^2\} \leq dw(X) \leq \sqrt{w^2(X) + \|X\|^4}$$

for any $X \in \mathcal{L}(\mathcal{H})$. It is easy to see that the Davis-Wielandt radius is not a norm. The Davis-Wielandt shell and its radius were first introduced and described by Davis [9] and Wielandt [29]. In fact, the Davis-Wielandt shell $DW(X)$ gives more information about the operator X and $W(X)$. For instance, in the finite-dimensional case, Li and Poon proved [26] that the normal property of Hilbert space operators can be completely determined by the geometrical shape of their Davis-Wielandt shells, namely, $X \in \mathcal{M}_n(\mathcal{C})$ is normal if and only if $DW(X)$ is a polyhedron in $\mathcal{C} \times \mathcal{R}$ identified with $\mathcal{R}^3$. Moreover, in the finite-dimensional case, the spectrum of an operator X ; $\mathbf{sp}(X)$ is finite, and $DW(X)$ is always closed. These conditions are no longer equivalent for an infinite-dimensional operator X . It has many properties, as discussed in reference [32]. The projection of the set $DW(X)$ on the first co-ordinate is $W(X)$. One can easily check that $dw(\cdot)$ is unitarily invariant, but it does not define a norm on $\mathcal{L}(\mathcal{H})$.

The Cartesian decomposition of an operator $X \in \mathcal{L}(\mathcal{H})$ can be written as $X = \Re(X) + i\Im(X)$, where $\Re(X) = \frac{X+X^*}{2}$ and $\Im(X) = \frac{X-X^*}{2i}$. A generalization of this decomposition was introduced in [28], called the weighted real and imaginary parts of X defined as:

$$\Re_\nu(X) = (1-\nu)X^* + \nu X \text{ \& \; } \Im_\nu(X) = \frac{(1-\nu)X - \nu X^*}{i} \quad \text{for all } \nu \in [0, 1].$$

Obviously, for $\nu = \frac{1}{2}$, $\Re_\nu(X) = \Re(X)$ and $\Im_\nu(X) = \Im(X)$. It is easy to see that every operator $X \in \mathcal{L}(\mathcal{H})$, $\Re_\nu(X) + i\Im_\nu(X) = (1-2\nu)X^* + X$. In [23], the authors defined the weighted numerical radius by the formula:

$$w_\nu(X) := \sup_{\|y\|=1} |(\Re_\nu(X) + i\Im_\nu(X))y, y| = w((1-2\nu)X^* + X)$$

for all $X \in \mathcal{L}(\mathcal{H})$ and $\nu \in [0, 1]$. Conde et al. [8] and Nayak [27] introduced the weighted numerical radius as follows:

$$w_\nu(X) := \sup_{\theta \in \mathcal{R}} \|\Re_\nu(e^{i\theta} X)\| \quad \text{for } \nu \in [0, 1].$$

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathcal{C}$ that the evaluation functionals $\varphi_\lambda(f) = f(\lambda)$, $\lambda \in \Theta$ are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem, there is a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$

for all $f \in \mathcal{H}$ and every $\lambda \in \Theta$. The function k on $\Theta \times \Theta$ defined by $k(z, \lambda) = k_\lambda(z)$ is called the reproducing kernel of $\mathcal{H}$ [2]. It was shown that $k_\lambda(z)$ can be represented by

$$k_\lambda(z) = \sum_{n=1}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $\{e_n\}_{n \geq 1}$ of $\mathcal{H}$, see [30]. For example, for the Hardy-Hilbert space $\mathcal{H}^2 = \mathcal{H}^2(\mathcal{D})$ over the unit disc $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\{z^n\}_{n \geq 1}$ is an orthonormal basis, and therefore the reproducing kernel of $\mathcal{H}^2$ is the function $k_\lambda(z) = \sum_{n=1}^{\infty} \overline{\lambda^n} z^n = (1 - \bar{\lambda}z)^{-1}$, $\lambda \in \mathcal{D}$.

Let $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For a given bounded linear operator X on $\mathcal{H}$, the Berezin symbol (or Berezin transform) of X is the bounded function $\widetilde{X}$ on Θ defined by

$$\widetilde{X}(\lambda) = \langle X \widehat{k}_\lambda(z), \widehat{k}_\lambda(z) \rangle, \lambda \in \Theta.$$

An important property of the Berezin symbol is that for all $X, Y \in \mathcal{L}(\mathcal{H})$ if $\widetilde{X}(\lambda) = \widetilde{Y}(\lambda)$ for all $\lambda \in \Theta$, then $X = Y$ (at least when $\mathcal{H}$ consists of analytic functions Zhu [33]). For more details, see [7, 11, 14, 17–22]. So, the map $X \rightarrow \widetilde{X}$ is injective [13]. The Berezin set and the Berezin number of an operator X are defined by $\mathbf{Ber}(X) = \{\widetilde{X}(\lambda) : \lambda \in \Theta\} = \text{Range}(\widetilde{X})$, and $\mathbf{ber}(X) = \sup\{|\gamma| : \gamma \in \mathbf{Ber}(X)\} = \sup_{\lambda \in \Theta} |\widetilde{X}(\lambda)|$, respectively.

The Crawford Berezin number of the operator X is defined by $c_{\mathbf{ber}}(X) := \inf\{|\widetilde{X}(\lambda)| : \lambda \in \Theta\}$ [18]. In [10], the authors defined the weighted Berezin radius and the weighted Berezin norm for all operators $X \in \mathcal{L}(\mathcal{H})$ and all $\nu \in [0, 1]$, as follows:

$$\mathbf{ber}_\nu(X) := \sup_{\lambda \in \Theta} |\langle \Re_\nu(X) + i\Im_\nu(X) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle|$$

and

$$\|X\|_{\mathbf{ber}_\nu} := \sup_{\lambda \in \Theta} \|(\Re_\nu(X) + i\Im_\nu(X)) \widehat{k}_\lambda\|.$$

The concepts of the Davis-Wielandt Berezin set and the Davis-Wielandt Berezin number were introduced in [1] as follows:

$$DW_{\mathbf{ber}}(X) = \{(\langle X \widehat{k}_\lambda, \widehat{k}_\lambda \rangle, \langle X \widehat{k}_\lambda, X \widehat{k}_\lambda \rangle) : \lambda \in \Theta\}$$

and

$$dw_{\mathbf{ber}}(X) = \sup_{\lambda \in \Theta} \left\{ \sqrt{|\langle X \widehat{k}_\lambda, \widehat{k}_\lambda \rangle|^2 + \|X \widehat{k}_\lambda\|^4} \right\}.$$

Now, we can see that $dw_{\mathbf{ber}}(X) \leq dw(X)$. It is easy to see that the Davis-Wielandt Berezin number of $X \in \mathcal{L}(\mathcal{H})$ satisfying the following inequality:

$$\max\{\mathbf{ber}(X), \|X\|_{\mathbf{ber}}^2\} \leq dw_{\mathbf{ber}}(X) \leq \sqrt{\mathbf{ber}^2(X) + \|X\|_{\mathbf{ber}}^4}. \quad (2)$$

Several properties and generalizations of the Berezin number and the Davis-Wielandt radius have been studied in [3, 4, 12, 15, 16, 31]. Utilizing the definition of the weighted

numerical radius, if $\mathcal{H} = \mathcal{H}(\Theta)$ is a reproducing kernel Hilbert space, for $X \in \mathcal{L}(\mathcal{H})$ and $\nu \in [0, 1]$, we define the weighted Davis-Wielandt Berezin number (WDWB) as follows:

$$dw_{\text{ber}_\nu}(X) := \sup \left\{ \sqrt{\left| (\Re_\nu(X) + i\Im_\nu(X))(\lambda) \right|^2 + \left\| (\Re_\nu(X) + i\Im_\nu(X))(\widehat{k}_\lambda) \right\|^4} : \lambda \in \Theta \right\}.$$

It is easy to see that this definition is equal to

$$dw_{\text{ber}_\nu}(X) := \sup \left\{ \sqrt{\left| ((1-2\nu)\widetilde{X^*} + X)(\lambda) \right|^2 + \left\| ((1-2\nu)X^* + X)(\widehat{k}_\lambda) \right\|^4} : \lambda \in \Theta \right\}.$$

By the definition of $dw_{\text{ber}_\nu}(\cdot)$, we get

$$\max \{ \text{ber}_\nu^2(X), \|X\|_{\text{ber}_\nu}^4 \} \leq dw_{\text{ber}_\nu}(X) \leq \sqrt{\text{ber}_\nu^2(X) + \|X\|_{\text{ber}_\nu}^4} \quad \text{for} \quad (3)$$

for all $X \in \mathcal{L}(\mathcal{H})$ and $\nu \in [0, 1]$.

In this paper, we study the weighted Davis-Wielandt Berezin number, deriving related inequalities and establishing several bounds for it.

2 Some inequalities for the weighted Davis-Wielandt Berezin number

In the present section, we show some new inequalities for the weighted Davis-Wielandt Berezin number of operators on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$. The following well-known lemmas are essential for proving our results. We begin with the Buzano inequality.

Lemma 1 [6] *Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e \rangle \langle e, x \rangle| \leq \frac{1}{2} (|\langle x, y \rangle| + \|x\| \|y\|).$$

The next lemma is the McCarty inequality for positive operators.

Lemma 2 (McCarty inequality). *Let $X \in \mathcal{L}(\mathcal{H})$ be a positive operator. Then, for all unit vectors $y \in \mathcal{H}$, we have*

$$\langle Xy, y \rangle^p \leq \langle X^p y, y \rangle,$$

where $p \geq 1$. This inequality is reversed if $0 < p \leq 1$.

The generalized mixed Schwarz inequality was introduced in [23], as follows:

Lemma 3 [23, Theorem 1] *Let $X \in \mathcal{L}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors.*

- *If f, g are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$), then $|\langle Xx, y \rangle| \leq \|f(|X|)x\| \|g(|X^*|)y\|$;*
- *If $0 \leq \alpha \leq 1$, then $|\langle Xx, y \rangle|^2 \leq \langle |X|^{2\alpha} x, x \rangle \langle |X^*|^{2(1-\alpha)} y, y \rangle$.*

Our first result of this section is lower and upper bounds for $dw_{\text{ber}_\nu}(\cdot)$.

Theorem 4 Let $X \in \mathcal{L}(\mathcal{H})$. Then

$$\begin{aligned} & \frac{1}{2} \left(\mathbf{ber}^2(X_v + |X_v|^2) + c_{\mathbf{ber}}^2(X_v - |X_v|^2) \right) \\ & \leq dw_{\mathbf{ber}_v}^2(X) \\ & \leq \frac{1}{2} \left(\mathbf{ber}^2(X_v + |X_v|^2) + \mathbf{ber}^2(X_v - |X_v|^2) \right) \end{aligned}$$

for all $v \in [0, 1]$ and $X_v = (1 - 2v)X^* + X$.

Proof If $\widehat{k}_\lambda \in \mathcal{H}(\Theta)$ and $v \in [0, 1]$, then

$$\begin{aligned} & \left| \langle ((1 - 2v)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left\| ((1 - 2v)X^* + X)\widehat{k}_\lambda \right\|^4 \\ &= \frac{1}{2} \left| \langle ((1 - 2v)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \langle ((1 - 2v)X^* + X)\widehat{k}_\lambda, ((1 - 2v)X^* + X)\widehat{k}_\lambda \rangle \right|^2 \\ & \quad + \frac{1}{2} \left| \langle ((1 - 2v)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle - \langle ((1 - 2v)X^* + X)\widehat{k}_\lambda, ((1 - 2v)X^* + X)\widehat{k}_\lambda \rangle \right|^2 \\ &= \frac{1}{2} \left| \langle ((1 - 2v)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \langle |(1 - 2v)X^* + X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \\ & \quad + \frac{1}{2} \left| \langle ((1 - 2v)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle - \langle |(1 - 2v)X^* + X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \\ &= \frac{1}{2} \left| \langle (1 - 2v)X^* + X + |(1 - 2v)X^* + X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \\ & \quad + \frac{1}{2} \left| \langle (1 - 2v)X^* + X - |(1 - 2v)X^* + X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \\ &\geq \frac{1}{2} \left(\left| \langle (1 - 2v)X^* + X + |(1 - 2v)X^* + X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \right. \\ & \quad \left. + c_{\mathbf{ber}}^2((1 - 2v)X^* + X - |(1 - 2v)X^* + X|^2) \right). \end{aligned}$$

By taking the supremum over all $\lambda \in \Theta$, we get

$$dw_{\mathbf{ber}_v}^2(X) \geq \frac{1}{2} \left(\mathbf{ber}^2(X_v + |X_v|^2) + c_{\mathbf{ber}}^2(X_v - |X_v|^2) \right),$$

where $v \in [0, 1]$ and $X_v = (1 - 2v)X^* + X$. To find the upper bound, we have

$$\begin{aligned} & \left| \langle ((1 - 2v)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left\| (1 - 2v)X^* + X\widehat{k}_\lambda \right\|^4 \\ &= \frac{1}{2} \left| \langle (1 - 2v)X^* + X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \langle (1 - 2v)X^* + X\widehat{k}_\lambda, (1 - 2v)X^* + X\widehat{k}_\lambda \rangle \right|^2 \\ & \quad + \frac{1}{2} \left| \langle (1 - 2v)X^* + X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle - \langle (1 - 2v)X^* + X\widehat{k}_\lambda, (1 - 2v)X^* + X\widehat{k}_\lambda \rangle \right|^2 \\ &= \frac{1}{2} \left| \langle (1 - 2v)X^* + X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \langle |(1 - 2v)X^* + X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \\ & \quad + \frac{1}{2} \left| \langle (1 - 2v)X^* + X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle - \langle |(1 - 2v)X^* + X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \\ &= \frac{1}{2} \left| \langle (1 - 2v)X^* + X + |X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \\ & \quad + \frac{1}{2} \left| \langle (1 - 2v)X^* + X - |X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \end{aligned}$$

$$\leq \frac{1}{2} \left(\mathbf{ber}^2((1-2\nu)X^* + X + |(1-2\nu)X^* + X|^2) \right. \\ \left. + \mathbf{ber}^2((1-2\nu)X^* + X - |(1-2\nu)X^* + X|^2) \right).$$

Similarly, by taking the supremum over all $\lambda \in \Theta$, we get

$$dw_{\mathbf{ber}_\nu}^2(X) \leq \frac{1}{2} \left(\mathbf{ber}^2(X_\nu + |X_\nu|^2) + \mathbf{ber}^2(X_\nu - |X_\nu|^2) \right),$$

where $X_\nu = (1-2\nu)X^* + X$. These statements complete the proof. $\square$

In the next theorem, we obtain some lower bounds for the weighted Davis-Wielandt Berezin number.

Theorem 5 *Let $X \in \mathcal{L}(\mathcal{H})$ and $\nu \in [0, 1]$. Then*

$$\max \left\{ \mathbf{ber}_\nu^2(X) + c_{\mathbf{ber}_\nu}^2(|X|^2), \|X\|_{\mathbf{ber}_\nu}^4 + c_{\mathbf{ber}_\nu}^2(X) \right\} \leq dw_{\mathbf{ber}_\nu}^2(X)$$

and

$$2 \max \left\{ \mathbf{ber}_\nu(X) c_{\mathbf{ber}_\nu}(|X|^2), \|X\|_{\mathbf{ber}_\nu}^2 c_{\mathbf{ber}_\nu}(X) \right\} \leq dw_{\mathbf{ber}_\nu}^2(X).$$

Proof If $\lambda \in \Theta$, then

$$\begin{aligned} & \left| \langle (1-2\nu)X^* + X, \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + c_{\mathbf{ber}}^2 \left(|(1-2\nu)X^* + X|^2 \right) \\ & \leq \left| \langle (1-2\nu)X^* + X, \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left| \langle (1-2\nu)X^* + X \rangle^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right|^2 \\ & = \left| \langle (1-2\nu)X^* + X, \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left\| (1-2\nu)X^* + X \right\|_{\widehat{k}_\lambda}^4 \\ & \leq dw_{\mathbf{ber}_\nu}^2(X). \end{aligned} \quad (4)$$

Similarly, we have

$$\begin{aligned} & c_{\mathbf{ber}}^2 \left((1-2\nu)X^* + X \right) + \left\| ((1-2\nu)X^* + X) \widehat{k}_\lambda \right\|^4 \\ & \leq \left| \langle ((1-2\nu)X^* + X) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left\| ((1-2\nu)X^* + X) \widehat{k}_\lambda \right\|^4 \\ & \leq dw_{\mathbf{ber}_\nu}^2(X). \end{aligned} \quad (5)$$

By taking the supremum over all $\lambda \in \Theta$ in inequalities (4) and (5), we conclude that the first inequality holds.

For the second inequality, by applying the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} & 2 \left| \langle ((1-2\nu)X^* + X) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| c_{\mathbf{ber}}(|(1-2\nu)X^* + X|^2) \\ & \leq 2 \left| \langle ((1-2\nu)X^* + X) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \left| \langle (1-2\nu)X^* + X \rangle^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right| \\ & = 2 \left| \langle ((1-2\nu)X^* + X) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \left\| (1-2\nu)X^* + X \right\|_{\widehat{k}_\lambda}^2 \end{aligned}$$

$$\begin{aligned} &\leq \left| \langle ((1-2\nu)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left\| ((1-2\nu)X^* + X)\widehat{k}_\lambda \right\|^4 \\ &\leq dw_{\text{ber}_\nu}^2(X). \end{aligned} \quad (6)$$

Likewise, we have

$$\begin{aligned} &2c_{\text{ber}}((1-2\nu)X^* + X) \left\| ((1-2\nu)X^* + X)\widehat{k}_\lambda \right\|^2 \\ &\leq 2 \left| \langle ((1-2\nu)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \left\| ((1-2\nu)X^* + X)\widehat{k}_\lambda \right\|^2 \\ &\leq dw_{\text{ber}_\nu}^2(X). \end{aligned} \quad (7)$$

Now, by taking the supremum over $\lambda \in \Theta$ in inequalities (6) and (7), we have the second result. $\square$

Remark 6 Note that the inequalities obtained in Theorem 5 (i) are sharper than the lower bound obtained in (3). In fact,

$$\begin{aligned} &\max \{ \text{ber}_\nu^2(X), \|X\|_{\text{ber}_\nu}^4 \} \\ &\leq \max \{ \text{ber}_\nu^2(X) + c_{\text{ber}_\nu}^2(|X|^2), \|X\|_{\text{ber}_\nu}^4 + c_{\text{ber}_\nu}^2(X) \} \\ &\leq dw_{\text{ber}_\nu}^2(X) \end{aligned}$$

for all $X \in \mathcal{L}(\mathcal{H})$ and $\nu \in [0, 1]$.

Corollary 7 [1, Theorem 3] Let $X \in \mathcal{L}(\mathcal{H})$. Then

$$\max \{ \text{ber}^2(X) + c_{\text{ber}}^2(|X|^2), \|X\|_{\text{ber}}^4 + c_{\text{ber}}^2(X) \} \leq dw_{\text{ber}}^2(X)$$

and

$$2 \max \{ \text{ber}(X) c_{\text{ber}}(|X|^2), \|X\|_{\text{ber}}^2 c_{\text{ber}}(X) \} \leq dw_{\text{ber}}^2(X).$$

Proof If we take $\nu = \frac{1}{2}$ in Theorem 5, then we get the desired result. $\square$

By applying the Buzano inequality, we present another upper bound for WDWB.

Theorem 8 Let $X \in \mathcal{L}(\mathcal{H})$ and $\nu \in [0, 1]$. Then

$$\begin{aligned} &dw_{\text{ber}_\nu}(X) \\ &\leq \sqrt{(1-2\nu)^2 \text{ber}^2(X) + (1-2\nu) \text{ber}(X^2) + (1-\nu) \text{ber}(X^*X + XX^*) + \text{ber}(|X_\nu|^4)}, \end{aligned}$$

where $X_\nu = (1-2\nu)X^* + X$.

Proof Let $\lambda \in \Theta$ be arbitrary. Then, first note that

$$\left| \langle ((1-2\nu)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \leq (1-2\nu) \left| \langle X^*\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| + \left| \langle X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|.$$

Hence

$$\begin{aligned}
& \left| \langle ((1-2\nu)X^* + X)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \\
& \leq (1-2\nu)^2 \left| \langle X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left| \langle X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + 2(1-2\nu) \left| \langle X^*\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \left| \langle X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \\
& \leq (1-2\nu)^2 \left| \langle X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left| \langle X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \left| \langle X^*\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \\
& \quad + (1-2\nu) \left[\left| \langle X\widehat{k}_\lambda, X^*\widehat{k}_\lambda \rangle \right| + \left\| X\widehat{k}_\lambda \right\| \left\| X^*\widehat{k}_\lambda \right\| \right] \\
& \quad \text{(using Lemmas 1 and 3)} \\
& \leq (1-2\nu)^2 \left| \langle X\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \frac{1}{2} \left| \langle (|X|^2 + |X^*|^2)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \\
& \quad + (1-2\nu) \left| \langle X^2\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| + \frac{1}{2} (1-2\nu) \left| \langle (|X|^2 + |X^*|^2)\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \\
& \quad \text{(using Lemma 2)} \\
& \leq (1-2\nu)^2 \mathbf{ber}^2(X) + (1-2\nu) \mathbf{ber}(X^2) + (1-\nu) \mathbf{ber}(X^*X + XX^*).
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
\left\| ((1-2\nu)X^* + X)\widehat{k}_\lambda \right\|^4 &= \left\langle ((1-2\nu)X^* + X)\widehat{k}_\lambda, ((1-2\nu)X^* + X)\widehat{k}_\lambda \right\rangle^2 \\
&= \left\langle |(1-2\nu)X^* + X|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2 \\
&\leq \left\langle |(1-2\nu)X^* + X|^4 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \quad \text{(using Lemma 2)} \\
&\leq \mathbf{ber} \left(|(1-2\nu)X^* + X|^4 \right).
\end{aligned}$$

Hence

$$\begin{aligned}
& dw_{\mathbf{ber}_\nu}(X) : \\
&= \sup_{\lambda \in \Theta} \left\{ \sqrt{\left| \langle (1-2\nu)X^* + X(\lambda) \rangle \right|^2 + \left\| (1-2\nu)X^* + X(\widehat{k}_\lambda) \right\|^4} \right\} \\
&\leq \sqrt{(1-2\nu)^2 \mathbf{ber}^2(X) + (1-2\nu) \mathbf{ber}(X^2) + (1-\nu) \mathbf{ber}(X^*X + XX^*) + \mathbf{ber}(|X_\nu|^4)},
\end{aligned}$$

where $X_\nu = (1-2\nu)X^* + X$. This proves the theorem. $\square$

Remark 9 If we take $\nu = \frac{1}{2}$ in Theorem 8, then we get

$$dw_{\mathbf{ber}}(X) \leq \sqrt{\frac{1}{2} \mathbf{ber}(X^*X + XX^*) + \mathbf{ber}(|X|^4)}.$$

Lemma 10 Let $X, Y \in \mathcal{L}(\mathcal{H})$ and $\nu \in [0, 1]$. Then

$$dw_{\mathbf{ber}_\nu}(\mu X + (1-\mu)Y) \leq \mu dw_{\mathbf{ber}_\nu}(X) + (1-\mu) dw_{\mathbf{ber}_\nu}(Y) \quad \text{for all } \mu \in [0, 1].$$

Proof First note that the functions $|\langle \cdot, s \rangle| : \mathcal{H} \rightarrow \mathcal{R}$ ($s \in \mathcal{H}$) and $\|\cdot\|^2 : \mathcal{H} \rightarrow \mathcal{R}$ are convex. Therefore, for all $v, \mu \in [0, 1]$ and all $\lambda \in \Theta$, we have

$$\begin{aligned} & \left| \left\langle \left[(1-v)(\mu X^* + (1-\mu)Y^*) + (\mu X + (1-\mu)Y) \right] \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \\ &= \left| \left\langle \left[\mu((1-v)X^* + X) + (1-\mu)((1-v)Y^* + Y) \right] \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \\ &\leq \left| \left\langle \mu((1-v)X^* + X) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \left| \left\langle (1-\mu)((1-v)Y^* + Y) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \\ &= \mu \left| \left\langle ((1-v)X^* + X) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + (1-\mu) \left| \left\langle ((1-v)Y^* + Y) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \end{aligned}$$

and

$$\begin{aligned} & \left\| \left[(1-v)(\mu X^* + (1-\mu)Y^*) + (\mu X + (1-\mu)Y) \right] \widehat{k}_\lambda \right\|^2 \\ &\leq \mu \left\| ((1-v)X^* + X) \widehat{k}_\lambda \right\|^2 + (1-\mu) \left\| ((1-v)Y^* + Y) \widehat{k}_\lambda \right\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} & dw_{\text{ber}_v}(\mu X + (1-\mu)Y) \\ &= \sup_{\lambda \in \Theta} \left\{ \left| \left\langle ((1-2v)(\mu X^* + (1-\mu)Y^*) + (\mu X + (1-\mu)Y) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle, \right. \right. \\ &\quad \left. \left. \left\| ((1-2v)(\mu X^* + (1-\mu)Y^*) + (\mu X + (1-\mu)Y) \widehat{k}_\lambda \right\|^2 \right) \right| \right\} \\ &\leq \sup_{\lambda \in \Theta} \left\{ \mu \left| \left\langle ((1-2v)X^* + X) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle, \left\| ((1-2v)X^* + X) \widehat{k}_\lambda \right\|^2 \right| \right. \\ &\quad \left. + (1-\mu) \left| \left\langle ((1-2v)Y^* + Y) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle, \left\| ((1-2v)Y^* + Y) \widehat{k}_\lambda \right\|^2 \right| \right\} \\ &\leq \mu \sup_{\lambda \in \Theta} \left| \left\langle ((1-2v)X^* + X) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle, \left\| ((1-2v)X^* + X) \widehat{k}_\lambda \right\|^2 \right| \\ &\quad + (1-\mu) \sup_{\lambda \in \Theta} \left| \left\langle ((1-2v)Y^* + Y) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle, \left\| ((1-2v)Y^* + Y) \widehat{k}_\lambda \right\|^2 \right| \\ &= \mu dw_{\text{ber}_v}(X) + (1-\mu) dw_{\text{ber}_v}(Y). \end{aligned}$$

□

Utilizing the Hermite-Hadamard inequality, we have the next result.

Proposition 11 *Let $X, Y \in \mathcal{L}(\mathcal{H})$ and $\lambda, v \in [0, 1]$. Then*

$$\begin{aligned} dw_{\text{ber}_v} \left(\frac{X+Y}{2} \right) &\leq k(\lambda) \\ &\leq \int_0^1 dw_{\text{ber}_v}(tX + (1-t)Y) dt \\ &\leq \frac{dw_{\text{ber}_v}(X) + dw_{\text{ber}_v}(Y)}{2}, \end{aligned}$$

where

$$k(\lambda) = (1 - \lambda)dw_{ber_v} \left(\frac{1 + \lambda}{2}X + \frac{1 - \lambda}{2}Y \right) + \lambda dw_{ber_v} \left(\frac{\lambda(X + Y)}{2} \right).$$

Proof The refined Hermite-Hadamard inequality for a convex function g on the interval $[0, 1]$ asserts that

$$g\left(\frac{1}{2}\right) \leq k(\lambda) \leq \int_0^1 g(t) dt \leq \frac{g(0) + g(1)}{2},$$

where

$$k(\lambda) = (1 - \lambda)g\left(\frac{1 + \lambda}{2}\right) + \lambda g\left(\frac{\lambda}{2}\right).$$

Now, utilizing the above inequality for the convex function $g(\mu) = dw_{ber_v}(\mu X + (1 - \mu)Y)$, we get the desired result. $\square$

Proposition 12 Let $X \in \mathcal{L}(\mathcal{H})$ be normal and $v \in [0, 1]$. Then

$$dw_{ber_v}(X) = dw_{ber_v}(X^*).$$

Proof First note that an operator T is normal if and only if $\|Ty\|^2 = \|T^*y\|^2$ for all $y \in \mathcal{H}$. Now, if X is normal operator, then $\Re_v(X) + i\Im_v(X) = (1 - 2v)X^* + X$ ($0 \leq v \leq 1$) is normal. Therefore,

$$\left\| (\Re_v(X) + i\Im_v(X))(\widehat{k}_\lambda) \right\|^2 = \left\| (\Re_v(X) - i\Im_v(X))(\widehat{k}_\lambda) \right\|^2 \quad \text{for all } \lambda \in \Theta.$$

Hence,

$$\begin{aligned} dw_{ber_v}^2(X) &= \sup_{\lambda \in \Theta} \left\{ \left| (\Re_v(X) + i\Im_v(X))(\widehat{k}_\lambda) \right|^2 + \left\| (\Re_v(X) + i\Im_v(X))(\widehat{k}_\lambda) \right\|^4 \right\} \\ &= \sup_{\lambda \in \Theta} \left\{ \left| (\Re_v(X) - i\Im_v(X))(\widehat{k}_\lambda) \right|^2 + \left\| (\Re_v(X) - i\Im_v(X))(\widehat{k}_\lambda) \right\|^4 \right\} \\ &= dw_{ber_v}^2(X^*). \end{aligned} \quad \square$$

Theorem 13 Let $X \in \mathcal{L}(\mathcal{H})$, $v \in [0, 1]$, and f, g are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, \infty)$). Then

$$dw_{ber_v}^2(X) \leq \text{ber} \left[\frac{1}{p} (f^{2p}(|X_v|) + f^{2p}(|X_v|^2)) + \frac{1}{q} (g^{2q}(|X_v^*|) + g^{2q}(|X_v|^2)) \right], \quad (8)$$

where $p \geq q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Utilizing Lemma 3, we have

$$\begin{aligned} &\left| (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda, \widehat{k}_\lambda \right|^2 + \left\| (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda \right\|^4 \\ &= \left| \langle (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \langle |\Re_v(X) + i\Im_v(X)|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^2 \end{aligned}$$

$$\begin{aligned}
&\leq \langle f^2 (|\Re_v(X) + i\Im_v(X)|) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \langle g^2 (|\Re_v(X) - i\Im_v(X)|) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \\
&\quad + \langle f^2 (|\Re_v(X) + i\Im_v(X)|^2) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \langle g^2 (|\Re_v(X) + i\Im_v(X)|^2) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \\
&\leq \frac{1}{p} \langle f^{2p} (|\Re_v(X) + i\Im_v(X)|) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^p + \frac{1}{q} \langle g^{2q} (|\Re_v(X) - i\Im_v(X)|) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^q \\
&\quad + \frac{1}{p} \langle f^{2p} (|\Re_v(X) + i\Im_v(X)|^2) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^p + \frac{1}{q} \langle g^{2q} (|\Re_v(X) + i\Im_v(X)|^2) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^q \\
&\leq \frac{1}{p} \langle f^{2p} (|\Re_v(X) + i\Im_v(X)|) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \frac{1}{q} \langle g^{2q} (|\Re_v(X) - i\Im_v(X)|) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \\
&\quad + \frac{1}{p} \langle f^{2p} (|\Re_v(X) + i\Im_v(X)|^2) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle + \frac{1}{2q} \langle g^{2q} (|\Re_v(X) + i\Im_v(X)|^2) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \\
&\leq \left\langle \frac{1}{p} (f^{2p} (|\Re_v(X) + i\Im_v(X)|^2) + f^{2p} (|\Re_v(X) + i\Im_v(X)|^2)) \right. \\
&\quad \left. + \frac{1}{q} (g^{2q} (|\Re_v(X) - i\Im_v(X)|^2) + g^{2q} (|\Re_v(X) + i\Im_v(X)|^2)) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\leq \mathbf{ber} \left[\frac{1}{p} (f^{2p} (|X_v|) + f^{2p} (|X_v|^2)) + \frac{1}{q} (g^{2q} (|X_v^*|) + g^{2q} (|X_v|^2)) \right],
\end{aligned}$$

where $X_v = \Re_v(X) + i\Im_v(X)$. Therefore, by taking the supremum over all $\lambda \in \Theta$, we get the desired result. $\square$

Corollary 14 *Let $X \in \mathcal{L}(\mathcal{H})$ and $v \in [0, 1]$. Then*

$$\begin{aligned}
&dw_{ber_v}^2(X) \\
&\leq \frac{1}{2} \mathbf{ber} ((|\Re_v(X) + i\Im_v(X)|^2 + |\Re_v(X) - i\Im_v(X)|^2 + 2|\Re_v(X) + i\Im_v(X)|^4)).
\end{aligned} \tag{9}$$

In particular,

$$dw_{ber}^2(X) \leq \frac{1}{2} \mathbf{ber} ((|X|^2 + |X^*|^2 + 2|X|^4)).$$

Proof If we put $f(t) = g(t) = t^{\frac{1}{2}}$ and $p = q = 2$ in inequality (8), then we immediately reach inequality (9). Moreover, if we take $v = \frac{1}{2}$, we get the second result. $\square$

Theorem 15 *Let $X \in \mathcal{L}(\mathcal{H})$. Then*

$$\begin{aligned}
&dw_{ber_v}^2(X) + 2c_{ber_v} ((\Re_v(X) + i\Im_v(X)) c_{ber_v} (|\Re_v(X) + i\Im_v(X)|^2) \\
&\leq \sup_{\theta \in \mathcal{R}} \mathbf{ber}^2 \left(e^{i\theta} (\Re_v(X) + i\Im_v(X)) + |\Re_v(X) + i\Im_v(X)|^2 \right).
\end{aligned}$$

Proof Let $\lambda \in \Theta$ and $\theta \in \mathcal{R}$. Then

$$\begin{aligned}
&\left| \langle (\Re_v(X) + i\Im_v(X)) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left| \langle \Re_v(X) + i\Im_v(X) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^4 \\
&\quad + 2 \left| \langle (\Re_v(X) + i\Im_v(X)) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \left| \langle \Re_v(X) + i\Im_v(X) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \\
&= \left| \langle e^{i\theta} (\Re_v(X) + i\Im_v(X)) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left| \langle \Re_v(X) + i\Im_v(X) \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^4
\end{aligned}$$

$$\begin{aligned}
& + 2 \left| \langle (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \left| \langle |\Re_v(X) + i\Im_v(X)|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \\
& = \left| \langle (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left\| (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda \right\|^4 \\
& \quad + 2 \left\langle e^{i\theta} (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left| \langle |\Re_v(X) + i\Im_v(X)|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \\
& = \left(\left\langle e^{i\theta} (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left| \langle |\Re_v(X) + i\Im_v(X)|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \right)^2 \\
& = \left\langle (e^{i\theta} (\Re_v(X) + i\Im_v(X)) + |\Re_v(X) + i\Im_v(X)|^2) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2 \\
& \leq \mathbf{ber}^2 \left(e^{i\theta} (\Re_v(X) + i\Im_v(X)) + |\Re_v(X) + i\Im_v(X)|^2 \right) \\
& \leq \sup_{\theta \in \mathcal{R}} \mathbf{ber}^2 \left(e^{i\theta} (\Re_v(X) + i\Im_v(X)) + |\Re_v(X) + i\Im_v(X)|^2 \right).
\end{aligned}$$

Hence

$$\begin{aligned}
& \left| \langle (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 + \left\| (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda \right\|^4 \\
& \leq \sup_{\theta \in \mathcal{R}} \mathbf{ber}^2 \left(e^{i\theta} (\Re_v(X) + i\Im_v(X)) + |\Re_v(X) + i\Im_v(X)|^2 \right) \\
& \quad - 2 \left| \langle (\Re_v(X) + i\Im_v(X))\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \left| \langle |\Re_v(X) + i\Im_v(X)|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \\
& \leq \sup_{\theta \in \mathcal{R}} \mathbf{ber}^2 \left(e^{i\theta} (\Re_v(X) + i\Im_v(X)) + |\Re_v(X) + i\Im_v(X)|^2 \right) \\
& \quad - 2c_{\mathbf{ber}_v}((\Re_v(X) + i\Im_v(X)) \ c_{\mathbf{ber}_v}(|\Re_v(X) + i\Im_v(X)|^2)).
\end{aligned}$$

Therefore, by taking the supremum over all $\lambda \in \Theta$, we get

$$\begin{aligned}
dw_{\mathbf{ber}_v}^2(X) & \leq \sup_{\theta \in \mathcal{R}} \mathbf{ber}^2 \left(e^{i\theta} (\Re_v(X) + i\Im_v(X)) + |\Re_v(X) + i\Im_v(X)|^2 \right) \\
& \quad - 2c_{\mathbf{ber}_v}((\Re_v(X) + i\Im_v(X)) \ c_{\mathbf{ber}_v}(|\Re_v(X) + i\Im_v(X)|^2)).
\end{aligned}$$

□

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