



Improved Berezin radius inequalities for certain operator matrices

Mojtaba Bakherad¹ · Fuad Kittaneh²

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Abstract

For a bounded linear operator T on a Hilbert space $\mathcal{H}$, the Berezin symbol (or Berezin transform) of T is the bounded function $\tilde{T}$ on a set Θ defined by $\tilde{T}(\tau) = \langle T\hat{k}_\tau(z), \hat{k}_\tau(z) \rangle$ for all $\tau \in \Theta$, where $\hat{k}_\tau = \frac{k_\tau}{\|k_\tau\|}$ is the normalized reproducing kernel of the space $\mathcal{H}$. The Berezin norm and the Berezin number (radius) of an operator T are defined, respectively, by $\|T\|_{\text{ber}} := \sup_{\tau, \mu \in \Theta} |\langle T\hat{k}_\tau, \hat{k}_\mu \rangle|$ and $\text{ber}(T) = \sup_{\tau \in \Theta} |\tilde{T}(\tau)|$. In this paper, we obtain some Berezin radius inequalities for certain operator matrices. Among other inequalities, it is shown that if A and B are bounded linear operators on a complex Hilbert space $\mathcal{H}$, then

$$\text{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \frac{1}{2} \sup_{\theta \in \mathbb{R}} \|e^{i\theta} A + e^{-i\theta} B^*\|_{\text{ber}} \leq \frac{1}{2} (\|A\|_{\text{ber}} + \|B\|_{\text{ber}}).$$

Keywords Berezin norm · Berezin radius · Operator matrix · Reproducing kernel

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1 Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the Banach algebra of all bounded linear operators defined on $\mathcal{H}$. In the case, when $\dim \mathcal{H} = n$, we identify $\mathcal{L}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field. For an operator $T \in \mathcal{L}(\mathcal{H})$, the Cartesian decomposition of T can be written as $T = \Re(T) + i\Im(T)$, where $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$.

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ such that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau \in \Theta$, are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem, there is a unique element $k_\tau \in \mathcal{H}$ such that

✉ Fuad Kittaneh
fkitt@ju.edu.jo

Mojtaba Bakherad
mojtaba.bakherad@gmail.com, bakherad@member.ams.org

¹ Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Islamic Republic of Iran

² Department of Mathematics, The University of Jordan, Amman, Jordan

$f(\tau) = \langle f, k_\tau \rangle$ for all $f \in \mathcal{H}$ and every $\tau \in \Theta$. The function k on $\Theta \times \Theta$ defined by $k(z, \tau) = k_\tau(z)$ is called the reproducing kernel of $\mathcal{H}$, see [1]. It was shown that $k_\tau(z)$ can be represented by

$$k_\tau(z) = \sum_{n=1}^{\infty} \overline{e_n(\tau)} e_n(z)$$

for any orthonormal basis $\{e_n\}_{n \geq 1}$ of $\mathcal{H}$. For example, for the Hardy-Hilbert space $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\{z^n\}_{n \geq 1}$ is an orthonormal basis. Therefore, the reproducing kernel of $\mathcal{H}^2$ is the function $k_\tau(z) = \sum_{n=1}^{\infty} \overline{\tau^n} z^n = (1 - \overline{\tau}z)^{-1}$, $\tau \in \mathbb{D}$.

Let $\hat{k}_\tau = \frac{k_\tau}{\|k_\tau\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$ (RKHS). For a given bounded linear operator T on $\mathcal{H}$, the Berezin symbol (or Berezin transform) of T is the bounded function $\tilde{T}$ on Θ defined by

$$\tilde{T}(\tau) = \left\langle T \hat{k}_\tau(z), \hat{k}_\tau(z) \right\rangle, \quad \tau \in \Theta.$$

An important property of the Berezin symbol is that for all $T, S \in \mathcal{L}(\mathcal{H})$, if $\tilde{T}(\tau) = \tilde{S}(\tau)$ for all $\tau \in \Theta$, then $T = S$ (at least when $\mathcal{H}$ consists of analytic functions, see Zhu [29]). For more details, see [3, 4, 13, 14, 17, 20, 21]. So, the map $T \rightarrow \tilde{T}$ is injective [16]. The Berezin set and the Berezin number (radius) of an operator T are defined, respectively, by

$$\mathbf{Ber}(T) = \mathbf{Range}(\tilde{T}) \quad \text{and} \quad \mathbf{ber}(T) = \sup_{\tau \in \Theta} |\tilde{T}(\tau)|.$$

In [3], the authors proved that

$$\mathbf{ber}(T) = \sup_{\theta \in \mathbb{R}} \mathbf{ber}(\Re(e^{i\theta} T)) \quad \text{for all } T \in \mathcal{L}(\mathcal{H}). \quad (1.1)$$

Using the above definition, it is easy to see that

$$\mathbf{ber}(T) = \sup_{\alpha^2 + \beta^2 = 1} \mathbf{ber}(\alpha \Re T + \beta \Im T),$$

where $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$. Recall that the numerical range and the numerical radius of $T \in \mathcal{L}(\mathcal{H})$ are defined, respectively, by

$$W(T) := \{\langle Tx, x \rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\} \quad \text{and} \quad w(T) := \sup\{|\lambda| : \lambda \in W(T)\}.$$

For more facts about the numerical radius, we refer the reader to [6, 11, 23–25] and references therein. It is clear that the Berezin transform $\tilde{A}$ is the bounded function on Θ whose values lie in the numerical range of the operator A , and hence $\mathbf{Ber}(A) \subseteq W(A)$ and $\mathbf{ber}(A) \leq w(A)$ for all $A \in \mathbb{B}(\mathcal{H})$. Karaev [22] showed that for $A = S \otimes S \in \mathbb{B}(\mathcal{H}^2)$, where S is the shift operator defined by $Sf(z) = zf(z)$ on the Hardy-Hilbert space $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we have $\tilde{A}(\lambda) = |\lambda|^2(1 - |\lambda|^2)$, and thus $\mathbf{Ber}(A) = [0, \frac{1}{4}] \subsetneq [0, 1] = W(A)$ and $\mathbf{ber}(A) = \frac{1}{4} < 1 = w(A)$. For more information about the Berezin number and the Berezin norm, see [5, 7, 9, 13, 15, 18, 19, 21, 27, 28] and references therein.

The Berezin norm of an operator $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$\|T\|_{\mathbf{ber}} := \sup\{|\langle T \hat{k}_\tau, \hat{k}_\mu \rangle| : \tau, \mu \in \Theta\}. \quad (1.2)$$

Clearly, by the Cauchy-Schwartz inequality, we have

$$\mathbf{ber}(T) \leq \|T\|_{\mathbf{ber}}. \quad (1.3)$$

For all positive operators $T \in \mathcal{L}(\mathcal{H})$, we have $\mathbf{ber}(T) = \|T\|_{\mathbf{ber}}$, see [10]. Moreover, for $T, S \in \mathcal{L}(\mathcal{H})$, it is clear from the above definitions of the Berezin radius (the Berezin number) and the Berezin norm that the following properties hold:

- (1) $\mathbf{ber}(\alpha T) = |\alpha| \mathbf{ber}(T)$ for all $\alpha \in \mathbb{C}$,
- (2) $\mathbf{ber}(T + S) \leq \mathbf{ber}(T) + \mathbf{ber}(S)$,
- (3) $\|\lambda T\|_{\mathbf{ber}} = |\lambda| \|T\|_{\mathbf{ber}}$ for all $\lambda \in \mathbb{C}$,
- (4) $\|T + S\|_{\mathbf{ber}} \leq \|T\|_{\mathbf{ber}} + \|S\|_{\mathbf{ber}}$,
- (5) $\|T\|_{\mathbf{ber}} = \|T^*\|_{\mathbf{ber}}$ and $\mathbf{ber}(T) = \mathbf{ber}(T^*)$.

For a Hilbert space $\mathcal{H}$, we denote the direct sum of 2-copies of $\mathcal{H}$ by $\mathcal{H} \oplus \mathcal{H}$. Due to this decomposition, any operator $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ can be represented as a 2×2 operator matrix of the form $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A, B, C, D \in \mathcal{L}(\mathcal{H})$. Further, if $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$, then $T\mathbf{x} = T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cx + Dy \end{bmatrix}$.

2 Some inequalities for the Berezin radius

The Berezin radius has several properties that are similar to the numerical radius. To learn about the numerical radii of operator matrices and their applications, we refer to [6, 23–25]. In contrast to the numerical radius, the Berezin radius is not weakly unitarily invariant, i.e., there exist an operator $A \in \mathcal{L}(\mathcal{H})$ and a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $\mathbf{ber}(U^*AU) \neq \mathbf{ber}(A)$. For instance, let $\{e_1, e_1\}$ be the standard orthonormal basis for $\mathbb{C}^2$ as a RKHS on the

set $\{1, 2\}$. If we put $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, then $\mathbf{ber}(U^*AU) = 2 \neq \mathbf{ber}(A) = 1$.

In the following proposition, we have a condition for the equality.

Proposition 2.1 *Let $\phi : \Theta \rightarrow \Theta$ be a bijective function. Then for all unitary operators $U \in \mathcal{L}(\mathcal{H})$ such that $U(\hat{k}_\tau) = \hat{k}_{\phi(\tau)}$ or $U(\hat{k}_\tau) = -\hat{k}_{\phi(\tau)}$ for all $\tau \in \Theta$, we have*

$$\mathbf{ber}(U^*AU) = \mathbf{ber}(A).$$

Proof By the definition of the Berezin radius, we have

$$\mathbf{ber}(U^*AU) = \sup_{\tau \in \Theta} \left| \left\langle U^*AU\hat{k}_\tau, \hat{k}_\tau \right\rangle \right| = \sup_{\tau \in \Theta} \left| \left\langle AU\hat{k}_\tau, U\hat{k}_\tau \right\rangle \right| = \sup_{\tau \in \Theta} \left| \left\langle A\hat{k}_{\phi(\tau)}, \hat{k}_{\phi(\tau)} \right\rangle \right| = \mathbf{ber}(A).$$

□

Example 2.2 Assume that $\{e_1, e_2, e_3\}$ is the standard orthonormal basis for $\mathbb{C}^3$ as a RKHS on the set $\Theta = \{1, 2, 3\}$. Let $\phi : \Theta \rightarrow \Theta$ be a function such that $\phi(1) = 3$, $\phi(2) = 1$

and $\phi(3) = 2$. For the unitary matrix $U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ such that $Ue_1 = e_3$, $Ue_2 = e_1$ and

$Ue_3 = e_2$, we have

$$\mathbf{ber}(U^*AU) = \max\{|a_{11}|, |a_{22}|, |a_{33}|\} = \mathbf{ber}(A)$$

for all matrices $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ in $\mathbb{M}_3$.

In the first lemma, we obtain some inequalities for 2×2 block-diagonal operator matrices. The second inequality in the following lemma was proven in [2]. Here, we give a new proof.

Lemma 2.3 *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then*

$$\mathbf{ber}\left(\frac{A+B}{2}\right) \leq \mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \max\{\mathbf{ber}(A), \mathbf{ber}(B)\}.$$

Proof For every $(\tau, \mu) \in \Theta \times \Theta$, let $\hat{\mathbf{k}}_{(\tau, \mu)} = \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H} \oplus \mathcal{H}$, i.e., $\|k_\tau\|^2 + \|k_\mu\|^2 = 1$. Then

$$\begin{aligned} \left| \langle (A+B)\hat{k}_\tau, \hat{k}_\tau \rangle \right| &= \left| \langle A\hat{k}_\tau, \hat{k}_\tau \rangle + \langle B\hat{k}_\tau, \hat{k}_\tau \rangle \right| \\ &= \left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\tau \end{bmatrix}, \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\tau \end{bmatrix} \right\rangle \right| \\ &\leq \sup \left\{ \left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\tau \end{bmatrix}, \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\tau \end{bmatrix} \right\rangle \right| : \tau \in \Theta \right\} \\ &= 2 \sup \left\{ \left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \frac{\hat{k}_\tau}{\sqrt{2}} \\ \frac{\hat{k}_\tau}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{\hat{k}_\tau}{\sqrt{2}} \\ \frac{\hat{k}_\tau}{\sqrt{2}} \end{bmatrix} \right\rangle \right| : \tau \in \Theta \right\} \\ &\leq 2 \sup \left\{ \left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}, \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix} \right\rangle \right| : \tau, \mu \in \Theta, \|k_\tau\|^2 + \|k_\mu\|^2 = 1 \right\} \\ &= 2\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right). \end{aligned}$$

Taking the supremum over $\tau \in \Theta$ in the above inequality, we have

$$\mathbf{ber}(A+B) \leq 2\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right).$$

Moreover, for $\hat{\mathbf{k}}_{(\tau, \mu)} = \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}$, we have

$$\begin{aligned} \left| \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}, \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix} \right\rangle \right| &= |\langle Ak_\tau, k_\tau \rangle + \langle Bk_\mu, k_\mu \rangle| \\ &\leq |\langle Ak_\tau, k_\tau \rangle| + |\langle Bk_\mu, k_\mu \rangle| \\ &\leq \mathbf{ber}(A)\|k_\tau\|^2 + \mathbf{ber}(B)\|k_\mu\|^2 \\ &\quad \text{(by the definition in (1.1))} \\ &\leq \max\{\mathbf{ber}(A), \mathbf{ber}(B)\}. \end{aligned}$$

The last inequality follows from the fact that

$$\sup_{\|k_\mu\|^2 + \|k_\tau\|^2 = 1} (x\|k_\tau\|^2 + y\|k_\mu\|^2) \leq \max\{x, y\} \quad \text{for all } x, y \geq 0.$$

By taking the supremum over $(\tau, \mu) \in \Theta \times \Theta$ in the above inequality, we obtain

$$\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \max\{\mathbf{ber}(A), \mathbf{ber}(B)\},$$

as required. $\square$

Corollary 2.4 Let $A \in \mathcal{L}(\mathcal{H})$. Then

- (1) $\mathbf{ber} \left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \right) = \mathbf{ber}(A)$.
 (2) $\mathbf{ber}(\Re(A)) \leq \mathbf{ber} \left(\begin{bmatrix} A & 0 \\ 0 & A^* \end{bmatrix} \right) \leq \mathbf{ber}(A)$.

Proof Applying $B = A$ in Lemma 2.3, we get the first result, and taking $B = A^*$ in Lemma 2.3, we have the second result. $\square$

Corollary 2.5 Let $A \in \mathcal{L}(\mathcal{H})$. Then

$$\mathbf{ber}(A) = \sup_{\theta \in \mathbb{R}} \mathbf{ber} \left(\begin{bmatrix} e^{i\theta} A & 0 \\ 0 & e^{-i\theta} A^* \end{bmatrix} \right).$$

Proof If we replace A by $e^{i\theta} A$ in Corollary 2.4(2), then

$$\mathbf{ber}(\Re(e^{i\theta} A)) \leq \mathbf{ber} \left(\begin{bmatrix} e^{i\theta} A & 0 \\ 0 & e^{-i\theta} A^* \end{bmatrix} \right) \leq \mathbf{ber}(e^{i\theta} A) \quad \text{for all } \theta \in \mathbb{R}.$$

It follows from $\mathbf{ber}(e^{i\theta} A) = \mathbf{ber}(A)$ and the definition in (1.1) that

$$\begin{aligned} \mathbf{ber}(A) &= \sup_{\theta \in \mathbb{R}} \mathbf{ber}(\Re(e^{i\theta} A)) \\ &\leq \sup_{\theta \in \mathbb{R}} \mathbf{ber} \left(\begin{bmatrix} e^{i\theta} A & 0 \\ 0 & e^{-i\theta} A^* \end{bmatrix} \right) \leq \mathbf{ber}(A). \end{aligned}$$

Hence,

$$\mathbf{ber}(A) = \sup_{\theta \in \mathbb{R}} \mathbf{ber} \left(\begin{bmatrix} e^{i\theta} A & 0 \\ 0 & e^{-i\theta} A^* \end{bmatrix} \right).$$

$\square$

In the following theorem, we find an estimate for $\mathbf{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right)$, where $A, B \in \mathcal{L}(\mathcal{H})$, see also [2].

Theorem 2.6 Let $A, B \in \mathcal{L}(\mathcal{H})$. Then

$$\mathbf{ber}(A + B) \leq 2\mathbf{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}}. \quad (2.1)$$

Proof Let $\hat{\mathbf{k}}_{(\tau, \mu)} = \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H} \oplus \mathcal{H}$. Then

$$\begin{aligned} \left| \left\langle \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}, \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix} \right\rangle \right| &= |\langle Ak_\mu, k_\tau \rangle + \langle Bk_\tau, k_\mu \rangle| \\ &\leq |\langle Ak_\mu, k_\tau \rangle| + |\langle Bk_\tau, k_\mu \rangle| \\ &\leq \|A\|_{\mathbf{ber}} \|k_\mu\| \|k_\tau\| + \|B\|_{\mathbf{ber}} \|k_\mu\| \|k_\tau\| \\ &\quad \text{(by the definition in (1.2))} \\ &= (\|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}}) \|k_\mu\| \|k_\tau\| \\ &\leq \frac{1}{2} (\|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}}). \end{aligned}$$

The last inequality follows from the fact that $\sup_{\|k_\mu\|^2 + \|k_\tau\|^2 = 1} \|k_\mu\| \|k_\tau\| = \frac{1}{2}$. Taking the supremum over $(\tau, \mu) \in \Theta \times \Theta$ in the above inequality, we get

$$2\mathbf{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}}.$$

Moreover, for $\tau \in \Theta$, we have

$$\begin{aligned} \left| \langle (A+B)\hat{k}_\tau, \hat{k}_\tau \rangle \right| &= \left| \langle A\hat{k}_\tau, \hat{k}_\tau \rangle + \langle B\hat{k}_\tau, \hat{k}_\tau \rangle \right| \\ &= \left| \left\langle \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\tau \end{bmatrix}, \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\tau \end{bmatrix} \right\rangle \right| \\ &\leq \sup_{\tau \in \Theta} \left| \left\langle \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\tau \end{bmatrix}, \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\tau \end{bmatrix} \right\rangle \right| \\ &\leq \sup_{\tau, \mu \in \Theta} \left| \left\langle \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\mu \end{bmatrix}, \begin{bmatrix} \hat{k}_\tau \\ \hat{k}_\mu \end{bmatrix} \right\rangle \right| \\ &= 2 \sup \left\{ \left| \left\langle \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} \frac{\hat{k}_\tau}{\sqrt{2}} \\ \frac{\hat{k}_\mu}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{\hat{k}_\tau}{\sqrt{2}} \\ \frac{\hat{k}_\mu}{\sqrt{2}} \end{bmatrix} \right\rangle \right| : \tau, \mu \in \Theta \right\} \\ &\leq 2 \sup \left\{ \left| \left\langle \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}, \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix} \right\rangle \right| : \tau, \mu \in \Theta, \|k_\tau\|^2 + \|k_\mu\|^2 = 1 \right\} \\ &= 2\mathbf{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right). \end{aligned}$$

Taking the supremum over $\tau \in \Theta$ in the above inequality, we have

$$\mathbf{ber}(A+B) \leq 2\mathbf{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right),$$

as required. $\square$

Remark 2.7 In [2], the authors proved that

$$2\mathbf{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \|A\| + \|B\| \quad \text{for all } A, B \in \mathcal{L}(\mathcal{H}). \quad (2.2)$$

It follows from $\|A\|_{\mathbf{ber}} \leq \|A\|$ and $\|B\|_{\mathbf{ber}} \leq \|B\|$ that the second inequality in (2.1) is a refinement of (2.2).

Remark 2.8 Consider $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ in the second inequality of (2.1). Then

$\mathbf{ber}^2 \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq 4$. Very recently, inspired by [12], the authors [8] showed another upper bound for the off-diagonal matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ as follows

$$\begin{aligned} \mathbf{ber}^2 \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) &\leq \frac{1}{|\alpha|} \max \{ \mathbf{ber}(AB), \mathbf{ber}(BA) \} \\ &\quad + \frac{\max \{1, |1-\alpha|\}}{2|\alpha|} \max \{ \|AA^* + B^*B\|_{\mathbf{ber}}, \|A^*A + BB^*\|_{\mathbf{ber}} \} \end{aligned} \quad (2.3)$$

for all $A, B \in \mathcal{L}(\mathcal{H})$ and all complex numbers $\alpha \in \mathbb{C} \setminus \{0\}$.

If we put $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ in the inequality (2.3), then we get

$$\mathbf{ber}^2 \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \frac{3}{|\alpha|} + 15 \frac{\max\{1, |1 - \alpha|\}}{2|\alpha|}$$

for all $\alpha \in \mathbb{C} \setminus \{0\}$. It follows from $\min_{\alpha \in \mathbb{R} \setminus \{0\}} \left(\frac{3}{|\alpha|} + 15 \frac{\max\{1, |1 - \alpha|\}}{2|\alpha|} \right) = \frac{21}{4}$ that the second inequality in (2.1) is sharper than the inequality (2.3) for $\alpha \in \mathbb{R} \setminus \{0\}$.

Example 2.9 Let $\{e_1, e_1\}$ be the standard orthonormal basis for $\mathbb{C}^2$ as a RKHS on the set $\{1, 2\}$. If we put the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ in Theorem 2.6, then we have

$$\mathbf{ber}(A + B) = 1 \leq 2\mathbf{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \approx 1.41 \leq \|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}} = 2.$$

In the following result, we have a refinement of (1.3).

Corollary 2.10 Let $A \in \mathcal{L}(\mathcal{H})$. Then

- (1) $\frac{1}{2}\mathbf{ber}(A) \leq \mathbf{ber} \left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right) \leq \|A\|_{\mathbf{ber}}$.
- (2) $\mathbf{ber}(\Re(A)) \leq \mathbf{ber} \left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \right) \leq \|A\|_{\mathbf{ber}}$.

Proof For part (1), it is enough to put $B = 0$ in Theorem 2.6, and for part (2), it is sufficient to put $B = A^*$ in Theorem 2.6. $\square$

Proposition 2.11 Let $A, B, S, T \in \mathcal{L}(\mathcal{H})$. Then

$$\mathbf{ber} \left(\begin{bmatrix} A & S \\ T & B \end{bmatrix} \right) = \mathbf{ber} \left(\begin{bmatrix} B & T \\ S & A \end{bmatrix} \right).$$

In particular,

- (1) $\mathbf{ber} \left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = \mathbf{ber} \left(\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} \right)$,
- (2) $\mathbf{ber} \left(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix} \right) = \mathbf{ber} \left(\begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} \right)$,
- (3) $\mathbf{ber} \left(\begin{bmatrix} A & S \\ 0 & 0 \end{bmatrix} \right) = \mathbf{ber} \left(\begin{bmatrix} 0 & 0 \\ S & A \end{bmatrix} \right)$.

Proof The proof follows from the fact that

$$\begin{aligned} & \mathbf{ber} \left(\begin{bmatrix} A & S \\ T & B \end{bmatrix} \right) \\ &= \sup \left\{ \left| \left\langle \begin{bmatrix} A & S \\ T & B \end{bmatrix} \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}, \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix} \right\rangle \right| : \tau, \mu \in \Theta, \|k_\tau\|^2 + \|k_\mu\|^2 = 1 \right\} \\ &= \sup \left\{ |\langle Ak_\tau, k_\tau \rangle + \langle Bk_\mu, k_\mu \rangle + \langle Sk_\mu, k_\tau \rangle + \langle Tk_\tau, k_\mu \rangle| : \tau, \mu \in \Theta, \|k_\tau\|^2 + \|k_\mu\|^2 = 1 \right\} \\ &= \sup \left\{ |\langle Ak_\mu, k_\mu \rangle + \langle Bk_\tau, k_\tau \rangle + \langle Sk_\tau, k_\mu \rangle + \langle Tk_\mu, k_\tau \rangle| : \tau, \mu \in \Theta, \|k_\tau\|^2 + \|k_\mu\|^2 = 1 \right\} \end{aligned}$$

$$\begin{aligned}
& \text{(by the symmetric relation between } \tau \text{ and } \mu) \\
&= \sup \left\{ \left| \left\langle \begin{bmatrix} B & T \\ S & A \end{bmatrix} \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}, \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix} \right\rangle \right| : \tau, \mu \in \Theta, \|k_\tau\|^2 + \|k_\mu\|^2 = 1 \right\} \\
&= \mathbf{ber} \left(\begin{bmatrix} B & T \\ S & A \end{bmatrix} \right),
\end{aligned}$$

as required. By taking $S = T = 0$, we have part (1), putting $A = B = 0$, we get part (2), and letting $B = T = 0$, we have part (3). $\square$

Remark 2.12 Notice that Proposition 2.11 also follows by applying Proposition 2.1. To see this, consider the bijective function $\phi : \Theta \times \Theta \rightarrow \Theta \times \Theta$ such that $\phi(\mu, \tau) = (\tau, \mu)$ and the unitary operator matrix $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ such that $U \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix} = \begin{bmatrix} k_\mu \\ k_\tau \end{bmatrix}$, where $\hat{\mathbf{k}}_{(\tau, \mu)} = \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}$ is the normalized reproducing kernel of $\mathcal{H} \oplus \mathcal{H}$. Then, using Proposition 2.1, we have

$$\mathbf{ber} \left(\begin{bmatrix} B & T \\ S & A \end{bmatrix} \right) = \mathbf{ber} \left(U^* \begin{bmatrix} B & T \\ S & A \end{bmatrix} U \right) = \mathbf{ber} \left(\begin{bmatrix} A & S \\ T & B \end{bmatrix} \right)$$

for all $A, B, S, T \in \mathcal{L}(\mathcal{H})$.

The following result shows another refinement of (1.3).

Corollary 2.13 *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

$$\mathbf{ber}(A) \leq \sup_{\theta \in \mathbb{R}} \mathbf{ber} \left(\begin{bmatrix} 0 & e^{i\theta} A \\ e^{-i\theta} A^* & 0 \end{bmatrix} \right) \leq \|A\|_{\mathbf{ber}}.$$

Proof Replacing A by $\frac{1}{2}e^{i\theta}A$ and B by $\frac{1}{2}e^{-i\theta}A^*$ in Theorem 2.6, respectively, we have

$$\begin{aligned}
\mathbf{ber} \left(\frac{1}{2}e^{i\theta}A + \frac{1}{2}e^{-i\theta}A^* \right) &\leq 2\mathbf{ber} \left(\begin{bmatrix} 0 & \frac{1}{2}e^{i\theta}A \\ \frac{1}{2}e^{-i\theta}A^* & 0 \end{bmatrix} \right) \\
&\leq \left\| \frac{1}{2}e^{i\theta}A \right\|_{\mathbf{ber}} + \left\| \frac{1}{2}e^{-i\theta}A^* \right\|_{\mathbf{ber}} \quad \text{for all } \theta \in \mathbb{R}.
\end{aligned}$$

It follows from $\|e^{i\theta}A\|_{\mathbf{ber}} = \|e^{-i\theta}A^*\|_{\mathbf{ber}} = \|A\|_{\mathbf{ber}}$ that

$$\mathbf{ber} \left(\Re(e^{i\theta}A) \right) \leq 2\mathbf{ber} \left(\begin{bmatrix} 0 & \frac{1}{2}e^{i\theta}A \\ \frac{1}{2}e^{-i\theta}A^* & 0 \end{bmatrix} \right) \leq \|A\|_{\mathbf{ber}} \quad \text{for all } \theta \in \mathbb{R}.$$

Hence, taking the supremum over $\theta \in \mathbb{R}$, we have

$$\mathbf{ber}(A) = \sup_{\theta \in \mathbb{R}} \mathbf{ber} \left(\Re(e^{i\theta}A) \right) \leq \sup_{\theta \in \mathbb{R}} \mathbf{ber} \left(\begin{bmatrix} 0 & e^{i\theta}A \\ e^{-i\theta}A^* & 0 \end{bmatrix} \right) \leq \|A\|_{\mathbf{ber}}.$$

This completes the proof. $\square$

In the following result, we have a refinement of the second inequality in (2.1).

Corollary 2.14 *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then*

$$\mathbf{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \frac{1}{2} \sup_{\theta \in \mathbb{R}} \|e^{i\theta}A + e^{-i\theta}B^*\|_{\mathbf{ber}} \leq \frac{1}{2} (\|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}}).$$

Proof Applying the definition in (1.1) and Theorem 2.6, we have

$$\begin{aligned}
 \text{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) &= \sup_{\theta \in \mathbb{R}} \text{ber} \left(\Re \left(e^{i\theta} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \right) \\
 &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \text{ber} \left(\begin{bmatrix} 0 & e^{i\theta} A + e^{-i\theta} B^* \\ e^{-i\theta} A^* + e^{i\theta} B & 0 \end{bmatrix} \right) \\
 &\leq \frac{1}{4} \sup_{\theta \in \mathbb{R}} \left(\|e^{i\theta} A + e^{-i\theta} B^*\|_{\text{ber}} + \|e^{-i\theta} A^* + e^{i\theta} B\|_{\text{ber}} \right) \\
 &\quad \text{(by the second inequality in (2.1))} \\
 &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \|e^{i\theta} A + e^{-i\theta} B^*\|_{\text{ber}} \quad (\text{since } \|T^*\|_{\text{ber}} = \|T\|_{\text{ber}}) \\
 &\leq \frac{1}{2} (\|A\|_{\text{ber}} + \|B\|_{\text{ber}}).
 \end{aligned}$$

□

Remark 2.15 Notice that if $A, B \in \mathcal{L}(\mathcal{H})$, then

$$\begin{aligned}
 |\langle (A+B)\hat{k}_\tau, \hat{k}_\mu \rangle| &\leq |\langle A\hat{k}_\tau, \hat{k}_\mu \rangle| + |\langle B\hat{k}_\tau, \hat{k}_\mu \rangle| \\
 &= \sup_{\theta \in \mathbb{R}} \left| e^{i\theta} \langle A\hat{k}_\tau, \hat{k}_\mu \rangle + e^{-i\theta} \langle B\hat{k}_\tau, \hat{k}_\mu \rangle \right| \\
 &= \sup_{\theta \in \mathbb{R}} \left| \langle (e^{i\theta} A + e^{-i\theta} B)\hat{k}_\tau, \hat{k}_\mu \rangle \right| \\
 &= \sup_{\theta \in \mathbb{R}} \|e^{i\theta} A + e^{-i\theta} B\|_{\text{ber}} \\
 &\leq \|A\|_{\text{ber}} + \|B\|_{\text{ber}} \quad \text{for all } \tau, \mu \in \Theta.
 \end{aligned}$$

Hence, by taking the supremum over $\tau, \mu \in \Theta$, we get

$$\|A+B\|_{\text{ber}} \leq \sup_{\theta \in \mathbb{R}} \|e^{i\theta} A + e^{-i\theta} B\|_{\text{ber}} \leq \|A\|_{\text{ber}} + \|B\|_{\text{ber}},$$

which is a refinement of the triangle inequality for $\|\cdot\|_{\text{ber}}$.

Theorem 2.16 Let $A, B \in \mathcal{L}(\mathcal{H})$. Then

$$2\text{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \sqrt{\|A+B^*\|_{\text{ber}}^2 + \|A-B^*\|_{\text{ber}}^2}. \quad (2.4)$$

Proof Let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Applying the definition of the Berezin radius, we have

$$\begin{aligned}
 \text{ber} \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) &= \sup_{\alpha^2 + \beta^2 = 1} \text{ber}(\alpha \Re(T) + \beta \Im(T)) \\
 &= \sup_{\alpha^2 + \beta^2 = 1} \text{ber} \left(\alpha \begin{bmatrix} 0 & \frac{A+B^*}{2} \\ \frac{A^*+B}{2} & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & \frac{A-B^*}{2i} \\ \frac{B-A^*}{2i} & 0 \end{bmatrix} \right) \\
 &\leq \sup_{\alpha^2 + \beta^2 = 1} \left[|\alpha| \text{ber} \left(\begin{bmatrix} 0 & \frac{A+B^*}{2} \\ \frac{A^*+B}{2} & 0 \end{bmatrix} \right) + |\beta| \text{ber} \left(\begin{bmatrix} 0 & \frac{A-B^*}{2i} \\ \frac{B-A^*}{2i} & 0 \end{bmatrix} \right) \right] \\
 &\leq \sup_{\alpha^2 + \beta^2 = 1} \left[|\alpha| \left\| \frac{A+B^*}{2} \right\|_{\text{ber}} + |\beta| \left\| \frac{A-B^*}{2i} \right\|_{\text{ber}} \right] \quad (\text{by Corollary 2.10(2)})
 \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\left\| \frac{A + B^*}{2} \right\|_{\text{ber}}^2 + \left\| \frac{A - B^*}{2i} \right\|_{\text{ber}}^2} \\
&= \frac{1}{2} \sqrt{\|A + B^*\|_{\text{ber}}^2 + \|A - B^*\|_{\text{ber}}^2},
\end{aligned}$$

as required. $\square$

Remark 2.17 Let $\{e_1, e_1\}$ be the standard orthonormal basis for $\mathbb{C}^2$. If we consider the 2×2 matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then

$$\sqrt{\|A + B^*\|_{\text{ber}}^2 + \|A - B^*\|_{\text{ber}}^2} \approx 2.82 \quad \text{and} \quad \|A\|_{\text{ber}} + \|B\|_{\text{ber}} = 2.$$

This shows that, in this example, the bound we have in Theorem 2.6 is better than that in Theorem 2.16.

However, if we take the matrices $A = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, then

$$\sqrt{\|A + B^*\|_{\text{ber}}^2 + \|A - B^*\|_{\text{ber}}^2} \approx 8.06 \quad \text{and} \quad \|A\|_{\text{ber}} + \|B\|_{\text{ber}} = 10.$$

Thus, neither Theorem 2.6 nor 2.16 is better than that the other.

Remark 2.18 Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\begin{aligned}
&\frac{1}{4} \left(\|A + B^*\|_{\text{ber}}^2 + \|A - B^*\|_{\text{ber}}^2 \right) \approx 1.23 \\
&\leq \min_{\alpha \in \mathbb{R} \setminus \{0\}} \left(\frac{1}{|\alpha|} + 3 \frac{\max\{1, |1 - \alpha|\}}{2|\alpha|} \right) \approx 1.56 \\
&\leq \frac{1}{|\alpha|} + 3 \frac{\max\{1, |1 - \alpha|\}}{2|\alpha|} \\
&= \frac{1}{|\alpha|} \max\{\text{ber}(AB), \text{ber}(BA)\} \\
&\quad + \frac{\max\{1, |1 - \alpha|\}}{2|\alpha|} \max\{\|AA^* + B^*B\|_{\text{ber}}, \|A^*A + BB^*\|_{\text{ber}}\}
\end{aligned}$$

for all $\alpha \in \mathbb{R} \setminus \{0\}$. Then, in this case, the inequality of (2.4) is sharper than the inequality (2.3).

In [2], the authors proved that

$$\text{ber}(A^*XB) \leq \frac{1}{2} \text{ber}(B^*|X|B + A^*|X^*|A), \quad (2.5)$$

where $A, B, X \in \mathcal{L}(\mathcal{H})$. Recently, the authors in [26], extended the inequality (2.5) as follows

$$\begin{aligned}
&\text{ber}(A^*XB + B^*YA) \\
&\leq \frac{1}{2} \text{ber}(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A + A^*|Y|^{2\alpha}A + B^*|Y^*|^{2(1-\alpha)}B) \\
&\quad - \frac{1}{4} \left(\frac{c^2(B^*|X|^{2\alpha}B - A^*|X^*|^{2(1-\alpha)}A)}{\text{ber}(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A)} + \frac{c^2(A^*|Y|^{2\alpha}A - B^*|Y^*|^{2(1-\alpha)}B)}{\text{ber}(A^*|Y|^{2\alpha}A + B^*|Y^*|^{2(1-\alpha)}B)} \right), \quad (2.6)
\end{aligned}$$

where $A, B, X, Y \in \mathcal{L}(\mathcal{H})$, $\alpha \in [0, 1]$ and $c(X) = \inf_{\mu \in \Theta} |\langle X \hat{k}_\mu, \hat{k}_\mu \rangle|$.

Using the inequality (2.5), Proposition 2.3(1) and Theorem 2.1, we have another extension of the inequality (2.5).

Theorem 2.19 *Let $A, B, C, D, X, Y \in \mathcal{L}(\mathcal{H})$. Then*

$$\begin{aligned} & \text{ber}(A^*XD + B^*YC) \\ & \leq \frac{1}{2} \max \left\{ \text{ber}(C^*|X|C + A^*|X^*|A), \text{ber}(B^*|Y^*|B + D^*|Y|D) \right\}. \end{aligned}$$

In particular,

$$\text{ber}(A^*D + B^*C) \leq \frac{1}{2} \max \left\{ \text{ber}(|C|^2 + |A|^2), \text{ber}(|B|^2 + |D|^2) \right\}.$$

Proof Consider the operator matrices $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $\hat{B} = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ and $\hat{X} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$. Then

$$\begin{aligned} & \text{ber}(A^*XD + B^*YC) \\ & \leq 2\text{ber} \left(\begin{bmatrix} 0 & A^*XD \\ B^*YC & 0 \end{bmatrix} \right) \quad (\text{by Theorem 2.1}) \\ & = 2\text{ber} \left(\begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \right) \\ & = 2\text{ber}(\hat{A}^* \hat{X} \hat{B}) \\ & \leq \text{ber}(\hat{B}^* |\hat{X}| \hat{B} + \hat{A}^* |\hat{X}^*| \hat{A}) \quad (\text{by the inequality (2.5)}) \\ & = \text{ber} \left(\begin{bmatrix} C^*|X|C + A^*|X^*|A & 0 \\ 0 & D^*|Y|D + B^*|Y^*|B \end{bmatrix} \right) \\ & \leq \frac{1}{2} \max \left\{ \text{ber}(C^*|X|C + A^*|X^*|A), \text{ber}(B^*|Y^*|B + D^*|Y|D) \right\} \\ & \quad (\text{by Proposition 2.3(1)}). \end{aligned}$$

This proves the first inequality. For the second inequality, put $X = Y = I$ in the first result. $\square$

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