



A New Family of Semi-Norms Between the Berezin Radius and the Berezin Norm

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Abstract

A functional Hilbert space is the Hilbert space $\mathcal{H}$ of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ such that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau \in \Theta$, are continuous on $\mathcal{H}$. The Berezin number of an operator X is defined by $\text{ber}(X) = \sup_{\tau \in \Theta} |\tilde{X}(\tau)| = \sup_{\tau \in \Theta} |\langle X\hat{k}_\tau, \hat{k}_\tau \rangle|$, where the operator X acts on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$ over some (non-empty) set Θ . In this paper, we introduce a new family involving means $\|\cdot\|_\sigma$ between the Berezin radius and the Berezin norm. Among other results, it is shown that if $X \in \mathcal{L}(\mathcal{H})$ and f, g are two non-negative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$, ($t \geq 0$), then

$$\|X\|_\sigma^2 \leq \text{ber} \left(\frac{1}{4} (f^4(|X|) + g^4(|X^*|)) + \frac{1}{2} |X|^2 \right)$$

and

$$\|X\|_\sigma^2 \leq \frac{1}{2} \sqrt{\text{ber}(f^4(|X|) + g^2(|X|^2)) \text{ber}(f^2(|X|^2) + g^4(|X^*|))},$$

where σ is a mean dominated by the arithmetic mean ∇ .

Keywords Reproducing kernel · Berezin number · Berezin transform · Berezin norm · Mean

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1 Introduction

Let $\mathcal{L}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ with the identity operator $I_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{H})$. When $\mathcal{H} = \mathbb{C}^n$, we identify $\mathcal{L}(\mathcal{H})$ with the algebra $\mathcal{M}_n(\mathbb{C})$ of n -by- n complex matrices.

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ such that the evaluation functionals $\varphi_{\tau}(f) = f(\tau)$, $\tau \in \Theta$, are continuous on $\mathcal{H}$. Then, by the Riesz representation theorem there is a unique element $k_{\tau} \in \mathcal{H}$ such that $f(\tau) = \langle f, k_{\tau} \rangle$ for all $f \in \mathcal{H}$ and every $\tau \in \Theta$. The function k on $\Theta \times \Theta$ defined by $k(z, \tau) = k_{\tau}(z)$ is called the reproducing kernel of $\mathcal{H}$, see [2, 4, 5, 17] and references therein. It was shown that $k_{\tau}(z)$ can be represented by

$$k_{\tau}(z) = \sum_{n=1}^{\infty} \overline{e_n(\tau)} e_n(z)$$

for any orthonormal basis $\{e_n\}_{n \geq 1}$ of $\mathcal{H}$, see [30]. For example, for the Hardy-Hilbert space $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\{z^n\}_{n \geq 1}$ is an orthonormal basis, therefore the reproducing kernel of $\mathcal{H}^2$ is the function $k_{\tau}(z) = \sum_{n=1}^{\infty} \overline{\tau^n} z^n = (1 - \overline{\tau}z)^{-1}$, $\tau \in \mathbb{D}$. Let $\widehat{k}_{\tau} = \frac{k_{\tau}}{\|k_{\tau}\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For a given a bounded linear operator X on $\mathcal{H}$, the Berezin symbol (or Berezin transform) of X is the bounded function $\widetilde{X}$ on Θ defined by

$$\widetilde{X}(\tau) = \langle X \widehat{k}_{\tau}(z), \widehat{k}_{\tau}(z) \rangle, \quad \tau \in \Theta.$$

An important property of the Berezin symbol is that for all $X, Y \in \mathcal{L}(\mathcal{H})$, if $\widetilde{X}(\tau) = \widetilde{Y}(\tau)$ for all $\tau \in \Theta$, then $X = Y$ (at least when $\mathcal{H}$ consists of analytic functions, see Zhu [31]). For more details, see [3, 6, 8–10, 12–14, 16, 18–29]. So, the map $X \rightarrow \widetilde{X}$ is injective [15]. The Berezin set and the Berezin number(radius) of an operator X are defined, respectively, by

$$\mathbf{Ber}(X) = \{\widetilde{X}(\tau) : \tau \in \Theta\} = \text{Range}(\widetilde{X})$$

and

$$\mathbf{ber}(X) = \sup\{|\gamma| : \gamma \in \mathbf{Ber}(X)\} = \sup_{\tau \in \Theta} |\widetilde{X}(\tau)|.$$

The Berezin norm of an operator $X \in \mathcal{L}(\mathcal{H})$ is defined by

$$\|X\|_{\mathbf{ber}} := \sup_{\tau \in \Theta} \|X \widehat{k}_{\tau}\|.$$

For $X, Y \in \mathcal{L}(\mathcal{H})$, it is clear from the above definitions of the Berezin radius (or the Berezin number) and the Berezin norm that the following properties hold:

- (1) $\mathbf{ber}(tX) = |t| \mathbf{ber}(X)$ for all $t \in \mathbb{C}$;
- (2) $\mathbf{ber}(X + Y) \leq \mathbf{ber}(X) + \mathbf{ber}(Y)$;
- (3) $\mathbf{ber}(X) \leq \|X\|_{\mathbf{ber}}$ and $\mathbf{ber}(X) = \mathbf{ber}(X^*)$;
- (4) $\|tX\|_{\mathbf{ber}} = |t| \|X\|_{\mathbf{ber}}$ for all $t \in \mathbb{C}$;
- (5) $\|X + Y\|_{\mathbf{ber}} \leq \|X\|_{\mathbf{ber}} + \|Y\|_{\mathbf{ber}}$.

In the recent paper [7], the authors defined the t -Berezin norm on $\mathcal{L}(\mathcal{H})$ as follows:

$$\|X\|_{t\text{-ber}} = \sup_{\tau \in \Theta} \sqrt{t|\tilde{X}(\tau)|^2 + (1-t)\|X\hat{k}_\tau\|^2}.$$

The t -Berezin norm is also a norm on $\mathcal{L}(\mathcal{H})$ for $t \in [0, 1)$, and for $t = 1$ it is a norm if the functional Hilbert space has the **Ber** property, i.e., for any two operators $X, Y \in \mathcal{L}(\mathcal{H})$ such that $\tilde{X}(\tau) = \tilde{Y}(\tau)$ for all $\tau \in \Theta$, we have $X = Y$. Hence, the t -Berezin norm is a norm in the familiar functional for Hilbert spaces, for instance Hardy and Bergman spaces. The t -Berezin norm satisfies the following inequalities:

$$\text{ber}(X) \leq \|X\|_{t\text{-ber}} \leq \|X\|_{\text{ber}} \quad \text{for } t \in [0, 1].$$

A binary function σ on $[0, +\infty)$ is called a mean, if the following conditions are satisfied:

- (i) If $a \leq b$, then $a \leq a \sigma b \leq b$;
- (ii) $a \leq c$ and $b \leq d$ imply $a \sigma b \leq c \sigma d$;
- (iii) σ is continuous in both variables;
- (iv) $t(a \sigma b) \leq (ta) \sigma (tb)$ ($t > 0$).

For instance, if $\mu \in (0, 1)$, the weighted geometric mean is $a \sharp_\mu b = a^{1-\mu} b^\mu$. The case $\mu = 1/2$ gives rise to the geometric mean $a \sharp b$. A mean σ is symmetric if $a \sigma b = b \sigma a$ for all positive numbers a, b . For a symmetric mean σ , a parametrized mean σ_t , $0 \leq t \leq 1$ is called an interpolational path for σ if it satisfies

- (1) $a \sigma_0 b = a$, $a \sigma_{1/2} b = a \sigma b$, and $a \sigma_1 b = b$;
- (2) $(a \sigma_p b) \sigma (a \sigma_q b) = a \sigma_{\frac{p+q}{2}} b$ for all $p, q \in [0, 1]$;
- (3) The map $t \in [0, 1] \mapsto a \sigma_t b$ is continuous for each a and b ;
- (4) σ_t is increasing in each of its components for $t \in [0, 1]$.

It is easy to see that the set of all $r \in [0, 1]$ satisfying

$$(a \sigma_p b) \sigma_r (a \sigma_q b) = a \sigma_{r p + (1-r) q} b \quad (1.1)$$

for all p, q is a convex subset of $[0, 1]$ including 0 and 1. For instance, the power means

$$a m_r b = \left(\frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \quad (r \in [-1, 1])$$

are some typical interpolational means. Their interpolational paths are

$$a m_{r,t} b = ((1-t)a^r + tb^r)^{\frac{1}{r}} \quad (t \in [0, 1]).$$

In particular, $a m_{1,t} b = a \nabla_t b = (1-t)a + tb$ is the weighted arithmetic mean, $a m_{0,t} b = a \sharp_t b = a^{1-t} b^t$ is the weighted geometric mean and $a m_{-1,t} b = a !_t b = ((1-t)a^{-1} + tb^{-1})^{-1}$ is the weighted harmonic mean. It is well-known that $a !_t b \leq a \sharp_t b \leq a \nabla_t b$ for positive numbers a and b and $t \in [0, 1]$. For more information about means, see [25] and references therein.

In this paper, we define a new quantity and establish some related results. The main ideas of this paper are stimulated by [7] and [11].

2 Main Results

We begin this section with the following definition.

Definition 2.1 Let $X \in \mathcal{L}(\mathcal{H})$ and σ_t be an interpolational path of a symmetric mean σ . We define

$$\|X\|_{\sigma_t} = \sup_{\tau \in \Theta} \left\{ \sqrt{|\tilde{X}(\tau)|^2 \sigma_t \|X\widehat{k}_\tau\|^2} \right\} \text{ for } 0 \leq t \leq 1.$$

Example 2.2 We consider the Hardy-Hilbert space, $\mathcal{H}^2(\mathbb{D})$, defined over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ as follows:

$$\mathcal{H}^2(\mathbb{D}) = \{f : \mathbb{D} \rightarrow \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

The inner product on $\mathcal{H}^2(\mathbb{D})$ is defined by $\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$, for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and $\{z^n\}_{n \geq 1}$ forms an orthonormal basis. We can identify $\mathcal{H}^2(\mathbb{D})$ with $l^2(\mathbb{N})$, since

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \leftrightarrow (a_0, a_1, a_2, \dots).$$

Therefore, the reproducing kernel of $\mathcal{H}^2(\mathbb{D})$ is given by the function $k_\tau(z) = \sum_{n=1}^{\infty} \overline{\tau}^n z^n = (1 - \overline{\tau}z)^{-1}$, $\tau \in \mathbb{D}$ and

$$\|k_\tau\|^2 = \langle k_\tau, k_\tau \rangle = k_\tau(\tau) = (1 - \overline{\tau}\tau)^{-1} = (1 - |\tau|^2)^{-1},$$

for any $\tau \in \mathbb{D}$.

On $l^2(\mathbb{N})$, we consider the unilateral shift operator U defined by

$$U(a_0, a_1, a_2, a_3, \dots) = (0, a_0, a_1, a_2, \dots)$$

for $(a_0, a_1, a_2, a_3, \dots) \in l^2(\mathbb{N})$. Thus, for any $\tau \in \mathbb{D}$, we have

$$\begin{aligned} |\widetilde{U}(\tau)|^2 &= |\langle U\widehat{k}_\tau(z), \widehat{k}_\tau(z) \rangle|^2 = \frac{1}{\|k_\tau\|^4} |\langle Uk_\tau(z), k_\tau(z) \rangle|^2 \\ &= \frac{1}{\|k_\tau\|^4} |\langle U(1, \overline{\tau}, \overline{\tau}^2, \dots), (1, \overline{\tau}, \overline{\tau}^2, \dots) \rangle|^2 \\ &= \frac{1}{\|k_\tau\|^4} |\langle (0, 1, \overline{\tau}, \overline{\tau}^2, \dots), (1, \overline{\tau}, \overline{\tau}^2, \dots) \rangle|^2 \\ &= \frac{1}{\|k_\tau\|^4} \left| \sum_{j=0}^{\infty} \overline{\tau}^j \tau^{j+1} \right|^2 = \frac{1}{\|k_\tau\|^4} \left(\frac{1}{1 - |\tau|^2} |\tau| \right)^2 = |\tau|^2, \end{aligned}$$

and due to the fact that U is an isometry, we conclude that

$$\|U\widehat{k}_\tau\|^2 = \|\widehat{k}_\tau\|^2 = 1.$$

Then, we obtain that

$$|\tilde{U}(\tau)|^2 \leq |\tau|^2 < 1 = \|U\widehat{k}_\tau\|^2, \quad (2.1)$$

and by the monotonicity of σ_t , we have

$$1 = \sup_{\tau \in \mathbb{D}} |\tau| = \sup_{\tau \in \mathbb{D}} |\tilde{U}(\tau)| \leq \sup_{\tau \in \mathbb{D}} \left\{ \sqrt{|\tilde{U}(\tau)|^2 \sigma_t \|U\widehat{k}_\tau\|^2} \right\} \leq \sup_{\tau \in \mathbb{D}} \|U\widehat{k}_\tau\| = 1 \quad (2.2)$$

for $t \in [0, 1]$.

In conclusion, we have that $\mathbf{ber}(U) \leq \|U\|_{\sigma_t} \leq \|U\|_{\mathbf{ber}}$, and in particular, $\|U\|_{\sigma_t} = 1$ for any $t \in [0, 1]$.

Following the ideas from the previous example and given that $|\tilde{X}(\tau)| \leq \|X\widehat{k}_\tau\|$, from the Cauchy-Schwartz inequality, we have that

$$\mathbf{ber}(X) \leq \|X\|_{\sigma_t} \leq \|X\|_{\mathbf{ber}}$$

for $t \in [0, 1]$ and $X \in \mathcal{L}(\mathcal{H})$. It is easy to see that for the special case $\sigma_t = \nabla_t$ ($0 \leq t \leq 1$), we have $\|\cdot\|_{\sigma_t} = \|\cdot\|_{(1-t)\text{-ber}}$.

The next proposition shows some properties of $\|\cdot\|_{\sigma_t}$.

Proposition 2.3 *Let $X \in \mathcal{L}(\mathcal{H})$ and σ_t, τ_μ be interpolational paths of symmetric means σ and τ . Then*

- (1) $\|X\|_{\sigma_0} = \mathbf{ber}(X)$ and $\|X\|_{\sigma_1} = \|X\|_{\mathbf{ber}}$;
- (2) $\|X\|_{\sigma_t} \leq \sqrt{\mathbf{ber}^2(X) \sigma_t \|X\|_{\mathbf{ber}}^2}$ for $t \in [0, 1]$;
- (3) $\|\tau X\|_{\sigma_t} = |\tau| \|X\|_{\sigma_t}$ for all $\tau \in \mathbb{C}$;
- (4) If the functional Hilbert space has the **Ber** property and $t \in [0, 1)$, then $\|X\|_{\sigma_t} = 0$ if and only if $X = 0$;
- (5) If $\sigma_t \leq \tau_s$, then $\|X\|_{\sigma_t} \leq \|X\|_{\tau_s}$ for $s, t \in [0, 1]$.

Remark 2.4 If $X \in \mathcal{L}(\mathcal{H})$, then

$$\begin{aligned} \| |X| \|_{\mathbf{ber}}^2 &= \sup_{\tau \in \Theta} \| |X| \hat{k}_\tau \|^2 = \sup_{\tau \in \Theta} \langle |X| \hat{k}_\tau, |X| \hat{k}_\tau \rangle \\ &= \sup_{\tau \in \Theta} \langle X^* X \hat{k}_\tau, \hat{k}_\tau \rangle = \sup_{\tau \in \Theta} \langle X \hat{k}_\tau, X \hat{k}_\tau \rangle = \sup_{\tau \in \Theta} \| X \hat{k}_\tau \|^2 = \| X \|_{\mathbf{ber}}^2 \end{aligned}$$

and for a semi-hyponormal operator X , i.e. $|X^*| \leq |X|$, the mixed Cauchy-Schwarz inequality $|\langle X \hat{k}_\tau, \hat{k}_\tau \rangle|^2 \leq \langle |X| \hat{k}_\tau, \hat{k}_\tau \rangle \langle |X^*| \hat{k}_\tau, \hat{k}_\tau \rangle$ implies that $|\langle X \hat{k}_\tau, \hat{k}_\tau \rangle|^2 \leq \langle |X| \hat{k}_\tau, \hat{k}_\tau \rangle^2$ for all $\hat{k}_\tau \in \mathcal{H}$, and then

$$\mathbf{ber}(X) = \sup_{\tau \in \Theta} |\langle X \hat{k}_\tau, \hat{k}_\tau \rangle| \leq \sup_{\tau \in \Theta} \langle |X| \hat{k}_\tau, \hat{k}_\tau \rangle = \mathbf{ber}(|X|).$$

Using the definition of $\|\cdot\|_{\sigma_t}$ and the monotonicity of σ_t , we have the next result.

Theorem 2.5 *Let $X \in \mathcal{L}(\mathcal{H})$ and σ_t be an interpolational path of a symmetric mean σ for all $t \in [0, 1]$. Then*

- (1) If X is hyponormal, i.e., $XX^* \leq X^*X$, then $\|X^*\|_{\sigma_t} \leq \|X\|_{\sigma_t}$.

- (2) If X is co-hyponormal, i.e., $X^*X \leq XX^*$, then $\|X\|_{\sigma_t} \leq \|X^*\|_{\sigma_t}$.
- (3) If X is semi-hyponormal, i.e., $|X^*| \leq |X|$, then $\|X\|_{\sigma_t} \leq \| |X| \|_{\sigma_t}$.
- (4) If X is (α, β) -normal, i.e., $\alpha X^*X \leq XX^* \leq \beta X^*X$ for some positive real numbers α and β with $\alpha \leq 1 \leq \beta$, then

$$\alpha \|X\|_{\sigma_t} \leq \|X^*\|_{\sigma_t} \leq \beta \|X\|_{\sigma_t}.$$

- (5) If X is normal, then $\|X^*\|_{\sigma_t} = \|X\|_{\sigma_t}$.

Proof (1) It follows from the hyponormality of X that $\|X^*\widehat{k}_\tau\| \leq \|X\widehat{k}_\tau\|$ for all $\tau \in \Theta$. Moreover, $|\langle X\widehat{k}_\tau, \widehat{k}_\tau \rangle| = |\langle X^*\widehat{k}_\tau, \widehat{k}_\tau \rangle|$ for all $\tau \in \Theta$. Hence, by the monotonicity of σ_t , we get

$$|\langle X^*\widehat{k}_\tau, \widehat{k}_\tau \rangle|^2 \sigma_t \|X^*\widehat{k}_\tau\|^2 \leq |\langle X\widehat{k}_\tau, \widehat{k}_\tau \rangle|^2 \sigma_t \|X\widehat{k}_\tau\|^2 \quad \text{for all } \tau \in \Theta.$$

Then, by the definition of $\|\cdot\|_{\sigma_t}$, we have $\|X^*\|_{\sigma_t} \leq \|X\|_{\sigma_t}$.

- (2) The proof is similar to that of part (1).
- (3) The condition of semi-hyponormality and the mixed Cauchy-Schwarz inequality imply that $|\langle X\widehat{k}_\tau, \widehat{k}_\tau \rangle|^2 \leq \langle |X|\widehat{k}_\tau, \widehat{k}_\tau \rangle^2$ for all $\widehat{k}_\tau \in \mathcal{H}$. Also, $\|X\widehat{k}_\tau\| = \| |X| \widehat{k}_\tau \|$ for all $\widehat{k}_\tau \in \mathcal{H}$. Therefore,

$$|\langle X\widehat{k}_\tau, \widehat{k}_\tau \rangle|^2 \sigma_t \|X\widehat{k}_\tau\|^2 \leq \langle |X|\widehat{k}_\tau, \widehat{k}_\tau \rangle^2 \sigma_t \| |X| \widehat{k}_\tau \|^2 \quad \text{for all } \tau \in \Theta.$$

By taking the supremum over all $\tau \in \Theta$, we get $\|X\|_{\sigma_t} \leq \| |X| \|_{\sigma_t}$.

- (4) Since X is (α, β) -normal, we have $\alpha \|X\widehat{k}_\tau\| \leq \|X^*\widehat{k}_\tau\| \leq \beta \|X\widehat{k}_\tau\|$ for all $\tau \in \Theta$. It follows from the fact that σ_t is increasing in its both variables that

$$\alpha^2 |\langle X\widehat{k}_\tau, \widehat{k}_\tau \rangle|^2 \sigma_t \alpha^2 \|X\widehat{k}_\tau\|^2 \leq |\langle X^*\widehat{k}_\tau, \widehat{k}_\tau \rangle|^2 \sigma_t \|X^*\widehat{k}_\tau\|^2 \leq \beta^2 |\langle X\widehat{k}_\tau, \widehat{k}_\tau \rangle|^2 \sigma_t \beta^2 \|X\widehat{k}_\tau\|^2$$

for all $\tau \in \Theta$. Hence, $\alpha \|X\|_{\sigma_t} \leq \|X^*\|_{\sigma_t} \leq \beta \|X\|_{\sigma_t}$.

- (5) It follows from the normality of X that X is both hyponormal and co-hyponormal, and then by the parts (1) and (2) we have the desired result. $\square$

Theorem 2.6 Let $X \in \mathcal{L}(\mathcal{H})$ and $t \in [0, 1]$. Then the following conditions are equivalent.

- (1) $\|X\|_{\sigma_t}^2 = \mathbf{ber}^2(X) \sigma_t \|X\|_{\mathbf{ber}}^2$.
- (2) There exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that

$$\lim_{n \rightarrow \infty} |\widetilde{X}(\tau_n)| = \mathbf{ber}(X) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|X\widehat{k}_{\tau_n}\| = \|X\|_{\mathbf{ber}}.$$

Proof We first prove that (1) implies (2). By the definition of the supremum, there exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that

$$\|X\|_{\sigma_t}^2 = \lim_{n \rightarrow \infty} |\widetilde{X}(\tau_n)|^2 \sigma_t \|X\widehat{k}_{\tau_n}\|^2.$$

It follows from the boundedness of the sequences $\{|\widetilde{X}(\tau_n)|\}$ and $\{\|X\widehat{k}_{\tau_n}\|\}$ that there exists a subsequence $\{\widehat{k}_{\tau_{n_k}}\}$ such that $\{|\widetilde{X}(\tau_{n_k})|\}$ and $\{\|X\widehat{k}_{\tau_{n_k}}\|\}$ are convergent. Then, we have

$$\mathbf{ber}^2(X) \sigma_t \|X\|_{\mathbf{ber}}^2 = \|X\|_{\sigma_t}^2$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} (|\tilde{X}(\tau_{nk})|^2 \sigma_t \|X \widehat{k}_{\tau_{nk}}\|^2) \\
&\leq \mathbf{ber}^2(X) \sigma_t \|X\|_{\mathbf{ber}}^2.
\end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} |\tilde{X}(\tau_{nk})| = \mathbf{ber}(X) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|X \widehat{k}_{\tau_{nk}}\| = \|X\|_{\mathbf{ber}}.$$

Now, we prove that (2) implies (1). We have

$$\begin{aligned}
\|X\|_{\sigma_t}^2 &= \sup_{\tau \in \Theta} \{|\tilde{X}(\tau)|^2 \sigma_t \|X \widehat{k}_\tau\|^2\} \\
&\geq \lim_{n \rightarrow \infty} \{|\tilde{X}(\tau_n)|^2 \sigma_t \|X \widehat{k}_{\tau_n}\|^2\} \\
&= \mathbf{ber}^2(X) \sigma_t \|X\|_{\mathbf{ber}}^2.
\end{aligned}$$

Hence, $\|X\|_{\sigma_t}^2 = \mathbf{ber}^2(X) \sigma_t \|X\|_{\mathbf{ber}}^2$. $\square$

We have seen in Proposition 2.3 that $\|\cdot\|_{\sigma_t}$ ($0 \leq t \leq 1$) fulfills the semi-norm properties, except possibly for the triangle inequality. In particular, when $\sigma = \nabla_t$ ($0 \leq t \leq 1$), we have the next proposition.

Proposition 2.7 *Let $X, Y \in \mathcal{L}(\mathcal{H})$ and $0 \leq t \leq 1$. Then*

$$\|X + Y\|_{\nabla_t} \leq \|X\|_{\nabla_t} + \|Y\|_{\nabla_t}.$$

Proof Let $\tau \in \Theta$ be a unit vector. Then

$$\begin{aligned}
&t|\widetilde{(X+Y)}(\tau)|^2 + (1-t)\|(X+Y)\widehat{k}_\tau\|^2 \\
&\leq t(|\tilde{X}(\tau)| + |\tilde{Y}(\tau)|)^2 + (1-t)(\|X\widehat{k}_\tau\| + \|Y\widehat{k}_\tau\|)^2 \\
&= t(|\tilde{X}(\tau)|^2 + |\tilde{Y}(\tau)|^2 + 2|\tilde{X}(\tau)||\tilde{Y}(\tau)|) \\
&\quad + (1-t)(\|X\widehat{k}_\tau\|^2 + \|Y\widehat{k}_\tau\|^2 + 2\|X\widehat{k}_\tau\|\|Y\widehat{k}_\tau\|) \\
&= t|\tilde{X}(\tau)|^2 + (1-t)\|X\widehat{k}_\tau\|^2 + t|\tilde{Y}(\tau)|^2 + (1-t)\|Y\widehat{k}_\tau\|^2 \\
&\quad + 2(t|\tilde{X}(\tau)||\tilde{Y}(\tau)| + (1-t)\|X\widehat{k}_\tau\|\|Y\widehat{k}_\tau\|).
\end{aligned}$$

Moreover, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
&t|\tilde{X}(\tau)||\tilde{Y}(\tau)| + (1-t)\|X\widehat{k}_\tau\|\|Y\widehat{k}_\tau\| \\
&\leq \sqrt{t|\tilde{X}(\tau)|^2 + (1-t)\|X\widehat{k}_\tau\|^2} \sqrt{t|\tilde{Y}(\tau)|^2 + (1-t)\|Y\widehat{k}_\tau\|^2}.
\end{aligned}$$

Combining the above inequalities, we have

$$\begin{aligned}
&t|\widetilde{(X+Y)}(\tau)|^2 + (1-t)\|(X+Y)\widehat{k}_\tau\|^2 \\
&\leq t|\tilde{X}(\tau)|^2 + (1-t)\|X\widehat{k}_\tau\|^2 + t|\tilde{Y}(\tau)|^2 + (1-t)\|Y\widehat{k}_\tau\|^2
\end{aligned}$$

$$\begin{aligned}
& + 2\sqrt{t|\widetilde{X}(\tau)|^2 + (1-t)\|X\widehat{k}_\tau\|^2}\sqrt{t|\widetilde{Y}(\tau)|^2 + (1-t)\|Y\widehat{k}_\tau\|^2} \\
& \leq \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2\|X\|_{\nabla_t}\|Y\|_{\nabla_t}.
\end{aligned}$$

Therefore,

$$\|X + Y\|_{\nabla_t}^2 = \sup_{\tau \in \Theta} \left\{ t|\widetilde{(X+Y)}(\tau)|^2 + (1-t)\|(X+Y)\widehat{k}_\tau\|^2 \right\} \leq (\|X\|_{\nabla_t} + \|Y\|_{\nabla_t})^2. \quad \square$$

In the following theorem, we give an equivalent condition that $\|X + Y\|_{\nabla_t} = \|X\|_{\nabla_t} + \|Y\|_{\nabla_t}$ for all $X, Y \in \mathcal{L}(\mathcal{H})$.

Theorem 2.8 *Let $X, Y \in \mathcal{L}(\mathcal{H})$ and $0 < t < 1$. Then the following conditions are equivalent.*

- (1) $\|X + Y\|_{\nabla_t} = \|X\|_{\nabla_t} + \|Y\|_{\nabla_t}$.
- (2) *There exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that*

$$\lim_{n \rightarrow \infty} \Re \left(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \right) = \|X\|_{\nabla_t}\|Y\|_{\nabla_t},$$

where $\Re(z)$ denotes the real part of a complex number z .

Proof (1) $\Rightarrow$ (2) Using the definition of the supremum and the hypothesis, there exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that

$$\lim_{n \rightarrow \infty} \left(t|\widetilde{(X+Y)}(\tau_n)|^2 + (1-t)\|(X+Y)\widehat{k}_{\tau_n}\|^2 \right) = (\|X\|_{\nabla_t} + \|Y\|_{\nabla_t})^2.$$

Hence,

$$\begin{aligned}
& t|\widetilde{(X+Y)}(\tau_n)|^2 + (1-t)\|(X+Y)\widehat{k}_{\tau_n}\|^2 \\
& = t(|\widetilde{X}(\tau_n)|^2 + |\widetilde{Y}(\tau_n)|^2 + 2\Re(\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n))) \\
& \quad + (1-t)(\|X\widehat{k}_{\tau_n}\|^2 + \|Y\widehat{k}_{\tau_n}\|^2 + 2\Re(\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle)) \\
& = t|\widetilde{X}(\tau_n)|^2 + (1-t)\|X\widehat{k}_{\tau_n}\|^2 + t|\widetilde{Y}(\tau_n)|^2 + (1-t)\|Y\widehat{k}_{\tau_n}\|^2 \\
& \quad + 2\Re(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) \\
& \leq \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2\Re(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) \\
& \leq \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2(t|\widetilde{X}(\tau_n)||\widetilde{Y}(\tau_n)| + (1-t)|\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle| |\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle|) \\
& \leq \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2(t\|X\widehat{k}_{\tau_n}\|\|Y\widehat{k}_{\tau_n}\| + (1-t)|\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle| |\langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle|) \\
& \leq \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2\sqrt{t|\widetilde{X}(\tau_n)|^2 + (1-t)\|X\widehat{k}_{\tau_n}\|^2}\sqrt{t|\widetilde{Y}(\tau_n)|^2 + (1-t)\|Y\widehat{k}_{\tau_n}\|^2} \\
& \quad \text{(by the Cauchy-Schwarz inequality)} \\
& \leq (\|X\|_{\nabla_t} + \|Y\|_{\nabla_t})^2.
\end{aligned}$$

Now, if we let $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \Re(t\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) = \|X\|_{\nabla_t}\|Y\|_{\nabla_t}.$$

(2) $\Rightarrow$ (1) Suppose that there exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that

$$\lim_{n \rightarrow \infty} \Re \left(t \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1-t) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \right) = \|X\|_{\nabla_t} \|Y\|_{\nabla_t}.$$

Then, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} & \Re^2 \left(t \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1-t) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \right) \\ & \leq \left| (1-t) \widetilde{X}(\tau_n) \widetilde{Y}(\tau_n) + t \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \right|^2 \\ & \quad - \Im^2 \left(t \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1-t) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \right) \\ & \leq \left| (1-t) \widetilde{X}(\tau_n) \widetilde{Y}(\tau_n) + t \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \right|^2, \end{aligned}$$

where $\Im(z)$ denotes the imaginary part of a complex number z . Hence,

$$\begin{aligned} & \Re \left(t \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1-t) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \right) \\ & \leq \left| (1-t) \widetilde{X}(\tau_n) \widetilde{Y}(\tau_n) + t \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \right| \\ & \leq (1-t) |\widetilde{X}(\tau_n)| |\widetilde{Y}(\tau_n)| + t |\langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle| |\langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle| \\ & \leq ((1-t) \|X \widehat{k}_{\tau_n}\| \|Y \widehat{k}_{\tau_n}\| + t |\langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle| |\langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle|) \\ & \leq \sqrt{t |\widetilde{X}(\tau_n)|^2 + (1-t) \|X \widehat{k}_{\tau_n}\|^2} \sqrt{t |\widetilde{Y}(\tau_n)|^2 + (1-t) \|Y \widehat{k}_{\tau_n}\|^2} \\ & \quad \text{(by the Cauchy-Schwarz inequality)} \\ & \leq \|X\|_{\nabla_t} \|Y\|_{\nabla_t}. \end{aligned} \tag{2.3}$$

It follows from $t |\widetilde{X}(\tau_n)|^2 + (1-t) \|X \widehat{k}_{\tau_n}\|^2 \leq \|X\|_{\nabla_t}^2$ and $t |\widetilde{Y}(\tau_n)|^2 + (1-t) \|Y \widehat{k}_{\tau_n}\|^2 \leq \|Y\|_{\nabla_t}^2$ that

$$\lim_{n \rightarrow \infty} \left(t |\widetilde{X}(\tau_n)|^2 + (1-t) \|X \widehat{k}_{\tau_n}\|^2 \right) = \|X\|_{\nabla_t}^2$$

and

$$\lim_{n \rightarrow \infty} \left(t |\widetilde{Y}(\tau_n)|^2 + (1-t) \|Y \widehat{k}_{\tau_n}\|^2 \right) = \|Y\|_{\nabla_t}^2.$$

Therefore,

$$\begin{aligned} (\|X\|_{\nabla_t} + \|Y\|_{\nabla_t})^2 &= \|X\|_{\nabla_t}^2 + \|Y\|_{\nabla_t}^2 + 2\|X\|_{\nabla_t} \|Y\|_{\nabla_t} \\ &= \lim_{n \rightarrow \infty} \left(t |\widetilde{X}(\tau_n)|^2 + (1-t) \|X \widehat{k}_{\tau_n}\|^2 \right) \\ & \quad + \lim_{n \rightarrow \infty} \left(t |\widetilde{Y}(\tau_n)|^2 + (1-t) \|Y \widehat{k}_{\tau_n}\|^2 \right) \\ & \quad + 2 \lim_{n \rightarrow \infty} \Re \left(t \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1-t) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \right) \\ &= \lim_{n \rightarrow \infty} \left[t (|\widetilde{X}(\tau_n)|^2 + |\widetilde{Y}(\tau_n)|^2) + 2 \Re (\widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n)) \right] \end{aligned}$$

$$\begin{aligned}
& + (1-t) \left(\|X\widehat{k}_{\tau_n}\|^2 + \|Y\widehat{k}_{\tau_n}\|^2 + 2\Re(\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) \right) \Big] \\
& = \lim_{n \rightarrow \infty} \left(t|\widetilde{(X+Y)}(\tau_n)|^2 + (1-t)\|(X+Y)\widehat{k}_{\tau_n}\|^2 \right) \\
& = \|X+Y\|_{\nabla_t}^2 \\
& \leq (\|X\|_{\nabla_t} + \|Y\|_{\nabla_t})^2 \\
& \quad \text{(by Proposition 2.7).}
\end{aligned}$$

Hence, $\|X+Y\|_{\nabla_t} = \|X\|_{\nabla_t} + \|Y\|_{\nabla_t}$. $\square$

Recently, Altwaijry et al. in [1], introduced the following generalization of $\|\cdot\|_{t-\text{ber}}$. Given non-negative real scalars α and β such that $(\alpha, \beta) \neq (0, 0)$ and $X \in \mathcal{L}(\mathcal{H})$, let

$$\|X\|_{\alpha, \beta}^{\text{ber}} = \sup_{\tau \in \Theta} \left\{ \sqrt{\beta|\widetilde{X}(\tau)|^2 + \alpha\|X\widehat{k}_{\tau}\|^2} \right\}. \quad (2.4)$$

Then, we have

$$\begin{aligned}
\frac{1}{\sqrt{\alpha+\beta}}\|X\|_{\alpha, \beta}^{\text{ber}} &= \sup_{\tau \in \Theta} \left\{ \sqrt{\frac{\beta|\widetilde{X}(\tau)|^2 + \alpha\|X\widehat{k}_{\tau}\|^2}{\alpha+\beta}} \right\} \\
&= \sup_{\tau \in \Theta} \left\{ \sqrt{\frac{\alpha}{\alpha+\beta}\|X\widehat{k}_{\tau}\|^2 + \frac{\beta}{\alpha+\beta}|\widetilde{X}(\tau)|^2} \right\} = \|X\|_{\nabla_{t_0}} = \|X\|_{\nabla_{t_1}}, \quad (2.5)
\end{aligned}$$

where $t_0 = \frac{\alpha}{\alpha+\beta}$ and $t_1 = \frac{\beta}{\alpha+\beta}$.

As a consequence of Theorem 2.8 and the previous identity, we derive the following characterization of the equality in the triangle inequality for the norm $\|\cdot\|_{\alpha, \beta}^{\text{ber}}$.

Corollary 2.9 [1, Theorem 10] *Let $X, Y \in \mathcal{L}(\mathcal{H})$ and non-negative real scalars α and β such that $(\alpha, \beta) \neq (0, 0)$. Then the following conditions are equivalent.*

- (1) $\|X+Y\|_{\alpha, \beta}^{\text{ber}} = \|X\|_{\alpha, \beta}^{\text{ber}} + \|Y\|_{\alpha, \beta}^{\text{ber}}$.
- (2) *There exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that*

$$\lim_{n \rightarrow \infty} (t_0\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t_0)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) = \|X\|_{\nabla_{t_0}}\|Y\|_{\nabla_{t_0}}, \quad (2.6)$$

where $t_0 = \frac{\alpha}{\alpha+\beta}$.

Proof We note that from the equality (2.5), the condition (1) is equivalent to

$$\|X+Y\|_{\nabla_{t_0}} = \|X\|_{\nabla_{t_0}} + \|Y\|_{\nabla_{t_0}}, \quad (2.7)$$

with $t_0 = \frac{\alpha}{\alpha+\beta}$. Now, by Theorem 2.8, the condition (2.7) is equivalent to the existence of a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that

$$\lim_{n \rightarrow \infty} \Re(t_0\widetilde{X^*}(\tau_n)\widetilde{Y}(\tau_n) + (1-t_0)\langle X\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y\widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) = \|X\|_{\nabla_{t_0}}\|Y\|_{\nabla_{t_0}}. \quad (2.8)$$

To finish the proof, it is enough to show that (2.8) is equivalent to

$$\lim_{n \rightarrow \infty} (t_0 \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1 - t_0) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) = \|X\|_{\nabla_{t_0}} \|Y\|_{\nabla_{t_0}}. \quad (2.9)$$

Indeed, if (2.9) holds, then

$$\begin{aligned} & \lim_{n \rightarrow \infty} (t_0 \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1 - t_0) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) \\ &= \|X\|_{\nabla_{t_0}} \|Y\|_{\nabla_{t_0}} = \frac{\|X\|_{\nabla_{t_0}} \|Y\|_{\nabla_{t_0}} + \overline{\|X\|_{\nabla_{t_0}} \|Y\|_{\nabla_{t_0}}}}{2} \\ &= \lim_{n \rightarrow \infty} \Re (t_0 \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1 - t_0) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle). \end{aligned} \quad (2.10)$$

On the other hand, by (2.3) for any $n \in \mathbb{N}$, we have

$$\begin{aligned} & \Re^2 (t_0 \widetilde{X}(\tau_n) \widetilde{Y}(\tau_n) + (1 - t_0) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) \\ & \leq (t_0 |\widetilde{X}(\tau_n)| |\widetilde{Y}(\tau_n)| + (1 - t_0) |\langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle| |\langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle|)^2 \\ & \leq \|X\|_{\nabla_{t_0}}^2 \|Y\|_{\nabla_{t_0}}^2. \end{aligned}$$

Thus, if we assume that condition (2.8) is fulfilled, then we can conclude that

$$\lim_{n \rightarrow \infty} \Im (t_0 \widetilde{X}(\tau_n) \widetilde{Y}(\tau_n) + (1 - t_0) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) = 0,$$

and

$$\lim_{n \rightarrow \infty} (t_0 \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1 - t_0) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) = \|X\|_{\nabla_{t_0}} \|Y\|_{\nabla_{t_0}},$$

and this completes the proof. $\square$

Remark 2.10 We note that condition (2.6) is equivalent to

(2') There exists a sequence $\{\widehat{k}_{\tau_n}\}$ in $\mathcal{H}$ such that

$$\lim_{n \rightarrow \infty} (\alpha \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + \beta \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) = \|X\|_{\alpha, \beta}^{\text{ber}} \|Y\|_{\alpha, \beta}^{\text{ber}}.$$

Indeed, if we denote by $t_0 = \frac{\alpha}{\alpha + \beta}$, then by (2.6)

$$\begin{aligned} \|X\|_{\alpha, \beta}^{\text{ber}} \|Y\|_{\alpha, \beta}^{\text{ber}} &= \sqrt{\alpha + \beta} \|X\|_{\nabla_{t_0}} \sqrt{\alpha + \beta} \|Y\|_{\nabla_{t_0}} \\ &= \lim_{n \rightarrow \infty} (\alpha + \beta) (t_0 \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + (1 - t_0) \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle) = \\ &= \lim_{n \rightarrow \infty} (\alpha \widetilde{X}^*(\tau_n) \widetilde{Y}(\tau_n) + \beta \langle X \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle \langle Y \widehat{k}_{\tau_n}, \widehat{k}_{\tau_n} \rangle). \end{aligned}$$

Finally, we remark that in [1, Theorem 10], the authors obtained a similar characterization of the equality $\|X + Y\|_{\alpha, \beta}^{\text{ber}} = \|X\|_{\alpha, \beta}^{\text{ber}} + \|Y\|_{\alpha, \beta}^{\text{ber}}$.

3 Some Estimations for $\|\cdot\|_{\sigma_t}$

In this section, we present some upper and lower bounds for $\|\cdot\|_{\sigma_t}$. The following well-known lemmas will be essential to prove our results.

Lemma 3.1 [25] *Let $X \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator with spectrum in an interval J and $\tau \in \Theta$.*

- (1) *If $f : J \rightarrow \mathbb{R}$ is convex, then $f(\langle X\widehat{k}_\tau, \widehat{k}_\tau \rangle) \leq \langle f(X)\widehat{k}_\tau, \widehat{k}_\tau \rangle$.*
- (2) *If $f : J \rightarrow \mathbb{R}$ is concave, then $f(\langle X\widehat{k}_\tau, \widehat{k}_\tau \rangle) \geq \langle f(X)\widehat{k}_\tau, \widehat{k}_\tau \rangle$.*

Lemma 3.2 [24] *Let $X \in \mathcal{L}(\mathcal{H})$ and f, g be two non-negative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$ for $t \geq 0$. Then*

$$|\widetilde{X}(\tau)| \leq |\widetilde{f^2(|X|)}(\tau)| |\widetilde{g^2(|X^*|)}(\tau)|$$

for all $\tau \in \Theta$.

For an operator $X \in \mathcal{L}(\mathcal{H})$, the Crawford Berezin number $\widetilde{c}(X)$ is defined as $\widetilde{c}(X) = \inf_{\tau \in \Theta} |\widetilde{X}(\tau)|$. In the following theorem, we obtain a lower bound for $\|\cdot\|_{\sigma_t}$ in the terms of $\widetilde{c}(\cdot)$.

Theorem 3.3 *Let $X \in \mathcal{L}(\mathcal{H})$ and $0 \leq t \leq 1$. Then,*

$$\|X\|_{\sigma_t} \geq \max \left\{ \sqrt{\mathbf{ber}^2(X)\sigma_t \widetilde{c}^2(X^*X)}, \sqrt{\widetilde{c}^2(X)\sigma_t \|X\|_{\mathbf{ber}}^2} \right\}.$$

Proof Let $\tau \in \Theta$ and $0 \leq t \leq 1$. Then

$$\begin{aligned} \|X\|_{\sigma_t}^2 &= \sup_{\tau \in \Theta} \{ |\widetilde{X}(\tau)|^2 \sigma_t \|X(\widehat{k}_\tau)\|^2 \} \\ &\geq |\widetilde{X}(\tau)|^2 \sigma_t \|X(\widehat{k}_\tau)\|^2 \\ &= |\widetilde{X}(\tau)|^2 \sigma_t |\widetilde{X^*X}(\tau)|^2 \\ &\geq |\widetilde{X}(\tau)|^2 \sigma_t \widetilde{c}^2(X^*X). \end{aligned}$$

Taking the supremum over all vectors $\tau \in \Theta$, we get $\|X\|_{\sigma_t}^2 \geq \mathbf{ber}^2(X)\sigma_t \widetilde{c}^2(X^*X)$. Similarly, we have

$$\|X\|_{\sigma_t}^2 = \sup_{\tau \in \Theta} \{ |\widetilde{X}(\tau)|^2 \sigma_t \|X(\widehat{k}_\tau)\|^2 \} \geq |\widetilde{X}(\tau)|^2 \sigma_t \|X(\widehat{k}_\tau)\|^2 \geq \widetilde{c}^2(X)\sigma_t \|X(\widehat{k}_\tau)\|^2,$$

whence $\|X\|_{\sigma_t}^2 \geq \widetilde{c}^2(X)\sigma_t \|X\|_{\mathbf{ber}}^2$. Combining the above inequalities, we get the desired result. $\square$

In the next result, we get some special case of Theorem 3.3.

Corollary 3.4 *Let $X \in \mathcal{L}(\mathcal{H})$, $r \in [-1, 1]$ and $0 \leq \mu \leq 1$. Then*

$$\|X\|_{m_{r,\mu}} \geq \max \left\{ \sqrt{\frac{(1-\mu)\mathbf{ber}^r(X) + \mu\widetilde{c}^r(X^*X)}{2}}, \sqrt{\frac{(1-\mu)\widetilde{c}^r(X) + \mu\|X\|_{\mathbf{ber}}^r}{2}} \right\}.$$

In particular,

$$\|X\|_{\nabla_\mu} \geq \max \left\{ \sqrt{(1-\mu)\mathbf{ber}^2(X) + \mu\widetilde{c}^2(X^*X)}, \sqrt{(1-\mu)\widetilde{c}^2(X) + \mu\|X\|_{\mathbf{ber}}^2} \right\}$$

and

$$\|X\|_{\sharp_\mu} \geq \max \left\{ \text{ber}^{(1-\mu)}(X) \widetilde{c}^\mu(X^*X), \widetilde{c}^{(1-\mu)}(X) \|X\|_{\text{ber}}^\mu \right\}.$$

Proof Letting σ_t be the interpolational paths of the power means $m_{r,\mu}$ for $r \in [-1, 1]$ and $0 \leq \mu \leq 1$ in Theorem 3.3, we have the first inequality. If we take the weighted arithmetic mean ∇_μ and the weighted geometric mean $\sharp_\mu$, ($0 \leq \mu \leq 1$) in Theorem 3.3, then we have the second and the third inequalities, respectively. $\square$

Theorem 3.5 Let $X \in \mathcal{L}(\mathcal{H})$, and let f, g be two non-negative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$ for $t \geq 0$. If σ is a mean dominated by the arithmetic mean ∇ , then

$$\|X\|_\sigma^2 \leq \text{ber} \left(\frac{1}{4} (f^4(|X|) + g^4(|X^*|)) + \frac{1}{2} |X|^2 \right)$$

and

$$\|X\|_\sigma^2 \leq \frac{1}{2} \sqrt{\text{ber}(f^4(|X|) + g^2(|X|^2)) \text{ber}(f^2(|X|^2) + g^4(|X^*|))}.$$

Proof Let $\tau \in \Theta$. Then

$$\begin{aligned} |\widetilde{X}(\tau)|^2 \sigma \|X \widehat{k}_\tau\|^2 &= |\widetilde{X}(\tau)|^2 \sigma |\widetilde{X^*X}(\tau)| \\ &\leq |\widetilde{f^2(|X|)}(\tau)| |\widetilde{g^2(|X^*|)}(\tau)| \sigma |\widetilde{X^*X}(\tau)| \\ &\quad \text{(by Lemma 3.2)} \\ &\leq \frac{1}{2} \left(|\widetilde{f^2(|X|)}(\tau)|^2 + |\widetilde{g^2(|X^*|)}(\tau)|^2 \right) \sigma |\widetilde{X^*X}(\tau)| \\ &\leq \frac{1}{2} \left(|\widetilde{f^4(|X|)}(\tau)| + |\widetilde{g^4(|X^*|)}(\tau)| \right) \sigma |\widetilde{X^*X}(\tau)| \\ &\quad \text{(by Lemma 3.1)} \\ &= \frac{1}{2} \left(|\widetilde{f^4(|X|) + g^4(|X^*|)}(\tau)| \right) \sigma |\widetilde{X^*X}(\tau)|. \end{aligned}$$

It follows from $\sigma \leq \nabla$ and the above inequalities that

$$\begin{aligned} |\widetilde{X}(\tau)|^2 \sigma \|X \widehat{k}_\tau\|^2 &\leq \frac{1}{2} \left(|\widetilde{f^4(|X|) + g^4(|X^*|)}(\tau)| \right) \sigma |\widetilde{X^*X}(\tau)| \\ &\leq \frac{1}{2} \left[\frac{1}{2} \left(|\widetilde{f^4(|X|) + g^4(|X^*|)}(\tau)| \right) + |\widetilde{X^*X}(\tau)| \right] \\ &\leq \left| \left(\frac{1}{4} (f^4(|X|) + g^4(|X^*|)) + \frac{1}{2} |X|^2 \right) (\tau) \right| \\ &\leq \text{ber} \left(\frac{1}{4} (f^4(|X|) + g^4(|X^*|)) + \frac{1}{2} |X|^2 \right). \end{aligned}$$

Then, by taking the supremum over $\tau \in \Theta$, we get the first result. For the second inequality, we have

$$\begin{aligned}
 |\tilde{X}(\tau)|^2 \sigma \|X \hat{k}_\tau\|^2 &= |\tilde{X}(\tau)|^2 \sigma |\tilde{X}|^2(\tau) \\
 &\leq |\widetilde{f^2(|X|)}(\tau)| |\widetilde{g^2(|X^*|)}(\tau)| \sigma \sqrt{|\widetilde{f^2(|X|^2)}(\tau)| |\widetilde{g^2(|X|^2)}(\tau)|} \\
 &= \sqrt{|\widetilde{f^2(|X|)}(\tau)|^2 |\widetilde{g^2(|X^*|)}(\tau)|^2 \sigma} \sqrt{|\widetilde{f^2(|X|^2)}(\tau)| |\widetilde{g^2(|X|^2)}(\tau)|} \\
 &\leq \sqrt{|\widetilde{f^4(|X|)}(\tau)| |\widetilde{g^4(|X^*|)}(\tau)| \sigma} \sqrt{|\widetilde{f^2(|X|^2)}(\tau)| |\widetilde{g^2(|X|^2)}(\tau)|} \\
 &\leq \frac{1}{2} \left(\sqrt{|\widetilde{f^4(|X|)}(\tau)| |\widetilde{g^4(|X^*|)}(\tau)|} + \sqrt{|\widetilde{f^2(|X|^2)}(\tau)| |\widetilde{g^2(|X|^2)}(\tau)|} \right) \\
 &\leq \frac{1}{2} \left(\sqrt{|\widetilde{f^4(|X|)}(\tau)| + |\widetilde{g^2(|X|^2)}(\tau)|} \sqrt{|\widetilde{f^2(|X|^2)}(\tau)| + |\widetilde{g^4(|X^*|)}(\tau)|} \right) \\
 &\quad \text{(by the Cauchy-Schwarz inequality } \sqrt{ab} + \sqrt{cd} \leq \sqrt{a+c} \sqrt{b+d} \text{)} \\
 &\leq \frac{1}{2} \left(\sqrt{|\widetilde{(f^4(|X|) + g^2(|X|^2))}(\tau)| |\widetilde{(f^2(|X|^2) + g^4(|X^*|))}(\tau)|} \right) \\
 &\leq \frac{1}{2} \sqrt{\mathbf{ber}(f^4(|X|) + g^2(|X|^2)) \mathbf{ber}(f^2(|X|^2) + g^4(|X^*|))}.
 \end{aligned}$$

Therefore,

$$\|X\|_\sigma^2 \leq \frac{1}{2} \sqrt{\mathbf{ber}(f^4(|X|) + g^2(|X|^2)) \mathbf{ber}(f^2(|X|^2) + g^4(|X^*|))}$$

For the special case $f(t) = g(t) = \sqrt{t}$, we have the following result.

Corollary 3.6 Let $X \in \mathcal{L}(\mathcal{H})$ and let σ is a mean dominated by the arithmetic mean ∇ . Then

$$\|X\|_\sigma^2 \leq \mathbf{ber} \left(\frac{3}{4} |X|^2 + \frac{1}{4} |X^*|^2 \right) \quad (3.1)$$

and

$$\|X\|_\sigma^2 \leq \frac{1}{2} \sqrt{2 \mathbf{ber}(|X|^2) \mathbf{ber}(|X|^2 + |X^*|^2)}.$$

Remark 3.7 If $X \in \mathcal{L}(\mathcal{H})$ and $\sigma \leq \nabla$, then by the inequality (3.1) and also the subadditivity of the Berezin radius, we have

$$\begin{aligned}
 \|X\|_\sigma^2 &\leq \mathbf{ber} \left(\frac{3}{4} |X|^2 + \frac{1}{4} |X^*|^2 \right) \\
 &\leq \frac{3}{4} \mathbf{ber}(|X|^2) + \frac{1}{4} \mathbf{ber}(|X^*|^2).
 \end{aligned}$$

Now, if X is normal, then by the definition of the Berezin radius, we get $\mathbf{ber}(|X^*|) = \mathbf{ber}(|X|)$. Hence, we have

$$\|X\|_\sigma^2 \leq \frac{3}{4} \mathbf{ber}(|X|^2) + \frac{1}{4} \mathbf{ber}(|X^*|^2)$$

$$\begin{aligned}
&\leq \mathbf{ber}(|X|^2) \\
&\leq \mathbf{ber}^2(|X|) \quad (\text{by Lemma 3.1})
\end{aligned} \tag{3.2}$$

Moreover, if X is positive, then the inequality (3.2) and the fact that $\mathbf{ber}(X) \leq \|X\|_\sigma$ imply that $\|X\|_\sigma = \mathbf{ber}(X)$.

Example 3.8 Consider for $\mathbb{C}^2$ the standard orthonormal basis $\{e_1, e_1\}$ as a RKHS on the set $\{1, 2\}$. Then for the self-adjoint matrix $X = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, which is not positive, we have

$$\mathbf{ber}(X) = 2 \not\leq \|X\|_\sigma = \sqrt{5}.$$

Theorem 3.9 Let $X \in \mathcal{L}(\mathcal{H})$, and let σ is a mean dominated by the arithmetic mean ∇ and $0 \leq \mu \leq 1$. Then

$$\|X\|_\sigma^2 \leq \frac{1}{2} \mathbf{ber}((1+\mu)|X|^2 + (1-\mu)|X^*|^2).$$

In particular,

$$\|X\|_\sigma^2 \leq \frac{1}{2} \mathbf{ber}(|X|^2 + |X^*|^2).$$

Proof Let $\tau \in \Theta$. Then

$$\begin{aligned}
|\tilde{X}(\tau)|^2 \sigma \|X \widehat{k}_\tau\|^2 &= |\tilde{X}(\tau)|^2 \sigma |\tilde{X}|^2(\tau)| \\
&\leq |\widehat{|X|^{2\mu}}(\tau)| |\widehat{|X^*|^{2(1-\mu)}}(\tau)| \sigma |\tilde{X}|^2(\tau)| \\
&\leq |\widehat{|X|^2}(\tau)|^\mu |\widehat{|X^*|^2}(\tau)|^{(1-\mu)} \sigma |\tilde{X}|^2(\tau)| \\
&\quad (\text{by Lemma 3.1}) \\
&\leq |(\mu|X|^2 + (1-\mu)|X^*|^2)(\tau)| \sigma |\tilde{X}|^2(\tau)| \\
&\quad (\text{by the weighted arithmetic-geometric mean inequality}) \\
&\leq \frac{1}{2} \left(|(\mu|X|^2 + (1-\mu)|X^*|^2)(\tau)| + |\tilde{X}|^2(\tau)| \right) \\
&= \frac{1}{2} \left(|((1+\mu)|X|^2 + (1-\mu)|X^*|^2)(\tau)| \right) \\
&\leq \frac{1}{2} \mathbf{ber}((1+\mu)|X|^2 + (1-\mu)|X^*|^2).
\end{aligned}$$

Taking the supremum over all $\tau \in \Theta$, we get

$$\|X\|_\sigma^2 \leq \frac{1}{2} \mathbf{ber}((1+\mu)|X|^2 + (1-\mu)|X^*|^2).$$

If we put $\mu = 0$, then we have the second inequality. □

Theorem 3.10 Let $X \in \mathcal{L}(\mathcal{H})$ and $0 \leq t \leq 1$. Then

$$\|X\|_{\nabla_t} \leq \inf_{\lambda \in [0,1]} \sqrt{\lambda \|X\|_{\text{ber}}^2 + (1-\lambda) \|X\|_{\text{ber}} \left((1-t) \|X\|_{\text{ber}} + t \text{ber}(X) \right)}.$$

Proof Given $u, v \in \mathcal{H}$ and $\lambda \in [0, 1]$, we have the following refinement of the classical Cauchy-Schwarz inequality:

$$\begin{aligned} |\langle u, v \rangle|^2 &= [(1-\lambda) + \lambda] |\langle u, v \rangle|^2 \\ &\leq (1-\lambda) |\langle u, v \rangle|^2 + \lambda \|u\|^2 \|v\|^2 \\ &\leq (1-\lambda) \|u\| \|v\| |\langle u, v \rangle| + \lambda \|u\|^2 \|v\|^2. \end{aligned} \quad (3.3)$$

Utilizing the inequality (3.3), yields

$$(1-t) |\langle u, v \rangle|^2 \leq (1-t)(1-\lambda) \|u\| \|v\| |\langle u, v \rangle| + (1-t)\lambda \|u\|^2 \|v\|^2, \quad (3.4)$$

and

$$t |\langle u, w \rangle|^2 \leq t(1-\lambda) \|u\| \|w\| |\langle u, w \rangle| + t\lambda \|u\|^2 \|w\|^2, \quad (3.5)$$

for all $u, v, w \in \mathcal{H}$ and $t \in [0, 1]$. Adding the relations (3.4), (3.5) and replacing v with $\frac{u}{\|v\|}$, we obtain

$$\begin{aligned} (1-t) \|u\|^2 + t |\langle u, w \rangle|^2 &\leq \lambda \|u\|^2 ((1-t) + t \|w\|^2) \\ &\quad + (1-\lambda) \|u\| ((1-t) \|u\| + t \|w\| |\langle u, w \rangle|). \end{aligned} \quad (3.6)$$

By substituting u for $X\hat{k}_\tau$ and w for $\hat{k}_\tau$, we have

$$\begin{aligned} (1-t) \|X\hat{k}_\tau\|^2 + t |\langle X\hat{k}_\tau, \hat{k}_\tau \rangle|^2 &\leq \lambda \|X\hat{k}_\tau\|^2 \\ &\quad + (1-\lambda) \|X\hat{k}_\tau\| \left((1-t) \|X\hat{k}_\tau\| + t |\langle X\hat{k}_\tau, \hat{k}_\tau \rangle| \right). \end{aligned} \quad (3.7)$$

Taking the supremum over all $\tau \in \Theta$, we have

$$\begin{aligned} \|X\|_{\nabla_t}^2 &= \sup_{\tau \in \Theta} \left\{ (1-t) \|X\hat{k}_\tau\|^2 + t |\langle X\hat{k}_\tau, \hat{k}_\tau \rangle|^2 \right\} \\ &\leq \sup_{\tau \in \Theta} \left\{ \lambda \|X\hat{k}_\tau\|^2 + (1-\lambda) \|X\hat{k}_\tau\| \left((1-t) \|X\hat{k}_\tau\| + t |\langle X\hat{k}_\tau, \hat{k}_\tau \rangle| \right) \right\} \\ &\leq \lambda \|X\|_{\text{ber}}^2 + (1-\lambda) \|X\|_{\text{ber}} \left((1-t) \|X\|_{\text{ber}} + t \text{ber}(X) \right), \end{aligned}$$

for any $\lambda \in [0, 1]$. □

Remark 3.11 Taking $\lambda = \frac{1}{3}$ in Theorem 3.10, we have a refinement of [1, Theorem 6]. Moreover, from the relation (2.5) and Theorem 3.10, we obtain for any pair of non-negative real numbers α, β such that $(\alpha, \beta) \neq (0, 0)$,

$$\|X\|_{\alpha, \beta}^{\text{ber}} = \sqrt{\alpha + \beta} \|X\|_{\nabla_{t_1}}$$

$$\begin{aligned} &\leq \sqrt{\alpha + \beta} \inf_{\lambda \in [0,1]} \sqrt{\lambda \|X\|_{\text{ber}}^2 + (1-\lambda) \|X\|_{\text{ber}} \left((1-t_1) \|X\|_{\text{ber}} + t_1 \mathbf{ber}(X) \right)} \\ &= \inf_{\lambda \in [0,1]} \sqrt{(\alpha + \beta)\lambda \|X\|_{\text{ber}}^2 + (1-\lambda) \|X\|_{\text{ber}} \left(\alpha \|X\|_{\text{ber}} + \beta \mathbf{ber}(X) \right)} \end{aligned}$$

where $t_1 = \frac{\beta}{\alpha + \beta}$.

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References

- Altwaijry, N., Feki, K., Minculete, N.: A generalized norm on reproducing kernel Hilbert spaces and its applications. *Axioms* **12**, 645 (2023)
- Aronzajn, N.: Theory of reproducing kernels. *Trans. Am. Math. Soc.* **68**, 337–404 (1950)
- Augustine, A., Garayev, M.T., Shankar, P.: Composition operators, convexity of their Berezin range and related questions. *Complex Anal. Oper. Theory* **17**(8), 126 (2023). 22 pp
- Bakherad, M.: Some Berezin number inequalities for operator matrices. *Czechoslov. Math. J.* **68**(143), 997–1009 (2018)
- Bakherad, M., Karaev, M.T.: Berezin number inequalities for Hilber space operators. *Concr. Oper.* **6**, 33–43 (2019)
- Bhunia, P., Garayev, M.T., Paul, K., Tapdigoglu, R.: Some new applications of Berezin symbols. *Complex Anal. Oper. Theory* **17**(6), 96 (2023). 15 pp
- Bhunia, P., Gürdal, M., Paul, K., Sen, A., Tapdigoglu, R.: On a new norm on the space of reproducing kernel Hilbert space operators and Berezin radius inequalities. *Numer. Funct. Anal. Optim.* **44**(9), 970–986 (2023)
- Bhunia, P., Paul, K., Sen, A.: Inequalities involving Berezin norm and Berezin number. *Complex Anal. Oper. Theory* **17**(1), 7 (2023). 15 pp
- Bhunia, P., Sen, A., Paul, K.: Development of the Berezin number inequalities. *Acta Math. Sin. Engl. Ser.* **39**(7), 1219–1228 (2023)
- Chien, F., Bakherad, M., Alomari, M.W.: Refined Berezin number inequalities via superquadratic and convex functions. *Filomat* **37**, 265–277 (2023)
- Conde, C., Moradi, H.R., Sababheh, M.: A family of semi-norms between the numerical radius and the operator norm. *Results Math.* **79**(1), 36 (2024). 17 pp
- Garayev, M.T.: On the invertibility of operators on a model space. *N.Y. J. Math.* **30**, 436–450 (2024)
- Garayev, M.T., Guesba, M.: Refinements of some inequalities involving Berezin norms and Berezin number and related questions. *Ann. Univ. Ferrara, Sez. 7: Sci. Mat.* **70**(2), 381–403 (2024)
- Garayev, M.T., Gürdal, M., Saltan, S.: Hardy type inequality for reproducing kernel Hilbert space operators and related problems. *Positivity* **21**(4), 1615–1623 (2017)
- Guillemin, V.: Toeplitz operators in n -dimensions. *Integral Equ. Oper. Theory* **7**(2), 145–205 (1984)
- Hajmohamadi, M., Lashkaripour, R., Bakherad, M.: Some generalizations of numerical radius on off-diagonal part of 2×2 operator matrices. *J. Math. Inequal.* **12**(2), 447–457 (2018)
- Hajmohamadi, M., Lashkaripour, R., Bakherad, M.: Improvements of Berezin number inequalities. *Linear Multilinear Algebra* **68**(6), 1218–1229 (2020)
- Karaev, M.T.: On the Berezin symbol. *Zap. Nauč. Semin. POMI* **270**, 80–89 (2000). (Russian); Translated from *Zapiski Nauchnykh Seminarov POMI* **115**(2), 2135–2140 (2003)

19. Karaev, M.T.: Berezin symbol and invertibility of operators on the functional Hilbert spaces. *J. Funct. Anal.* **238**(1), 181–192 (2006)
20. Karaev, M.T.: Reproducing kernels and Berezin symbols techniques in various questions of operator theory. *Complex Anal. Oper. Theory* **7**(4), 983–1018 (2013)
21. Karaev, M.T., Gürdal, M.: On the Berezin symbols and Toeplitz operators. *Extr. Math.* **25**(1), 83–102 (2010)
22. Karaev, M.T., Saltan, S.: Some results on Berezin symbols. *Complex Var.* **50**(3), 185–193 (2005)
23. Karaev, M.T., Gürdal, M., Huban, M.: Reproducing kernels, Engliš algebras and some applications. *Stud. Math.* **232**(2), 113–141 (2016)
24. Kittaneh, F.: Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24**(2), 283–293 (1988)
25. Pecaric, J., Furuta, T., Hot, J.M., Seo, Y.: *Mond-Pecaric Method in Operator Inequalities*. Element, Zagreb (2005)
26. Sain, D., Bhunia, P., Bhanja, A., Paul, K.: On a new norm on $B(H)$ and its applications to numerical radius inequalities. *Ann. Funct. Anal.* **12**(4), 51 (2021). 25 pp
27. Sen, A., Paul, K.: Berezin number and numerical radius inequalities. *Vietnam J. Math.* (2023). <https://doi.org/10.1007/s10013-023-00658-8>
28. Sen, A., Bhunia, P., Paul, K.: Bounds for the Berezin number of reproducing kernel Hilbert space operators. *Filomat* **37**(6), 1741–1749 (2023)
29. Tapdigoglu, R., Altwaijry, N., Garayev, M.T.: New inequalities via Berezin symbols and related questions. *Korean J. Math.* **31**(1), 109–120 (2023)
30. Yamancı, U., Garayev, M.T.: Some results related to the Berezin number inequalities. *Turk. J. Math.* **43**, 1940–1952 (2019)
31. Zhu, K.: *Operator Theory in Function Spaces*, 2nd edn. Dekker, New York (2007)

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