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GENERALIZED A -NUMERICAL RADIUS INEQUALITIES IN SEMI-HILBERT SPACES THROUGH THE ANGLE BETWEEN TWO VECTORS

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ABSTRACT. This paper presents several inequalities related to the A -numerical radius in the context of semi-Hilbert space operators. By integrating modern techniques from operator theory and functional analysis, we derive new inequalities for the A -numerical radius that emphasize the unique characteristics of semi-Hilbert space operators. Among other results, it is shown that, if $A, T \in \mathbb{B}(\mathcal{H})$, where A is a positive and onto operator, and T has a polar decomposition given by $T = U|T|$, $x \in \mathcal{H}$, $0 \leq \alpha \leq 1$ and $\theta_{T,x} = \angle_{|T|^{\alpha}x, |T|^{1-\alpha}U^*Ax}$, then

$$w_A(T) \leq \frac{\mu(\theta)}{2} \left\| |T|^{\alpha}U^*A|T|^{\alpha} + U|T|^{1-\alpha}|T|^{1-\alpha}U^*A \right\|_A,$$

where either $0 \leq \theta < \theta_{T,x} \leq \frac{\pi}{2}$ for all unit vectors $x \in \mathcal{H}$, or $\frac{\pi}{2} \leq \theta_{T,x} < \theta \leq \pi$ for all A -unit vectors $x \in \mathcal{H}$.

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Key words: A -numerical radius, semi-Hilbert space, angle, Cauchy-Schwarz inequality.

1. Introduction. Let $\mathbb{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ with the identity operator $I_{\mathcal{H}}$ in $\mathbb{B}(\mathcal{H})$. An operator $T \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and we then write $T \geq 0$. The absolute value of T is denoted by $|T|$, that is $|T| = (T^*T)^{\frac{1}{2}}$. For a bounded linear operator T on a Hilbert space $\mathcal{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \rightarrow \langle Tx, x \rangle$ associated with the operator. More precisely, $W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$. Also, the numerical radius is defined by $w(T) = \sup \{|\langle Tx, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}$. This norm is equivalent to the operator norm. In fact, the following inequalities hold

$$\frac{1}{2} \|T\| \leq \omega(T) \leq \|T\|.$$

Let $T \in \mathbb{B}(\mathcal{H})$. In [18], the author proved that

$$\frac{1}{4} \| |T|^2 + |T^*|^2 \| \leq \omega^2(T) \leq \frac{1}{2} \| |T|^2 + |T^*|^2 \|. \quad (1.1)$$

The second inequality in (1.1) is a special case of the following more general form [14]

$$\omega^{2r}(T) \leq \frac{1}{2} \| |T|^{2r} + |T^*|^{2r} \| \quad \text{for all } r \geq 1. \quad (1.2)$$

In [2], the authors proved that

$$\omega^{2r}(T) \leq \frac{1}{2} \| |T|^{2r} + \frac{1}{4} \| |T|^{4\alpha r} + |T^*|^{4(1-\alpha)r} \| \quad \text{for all } 0 \leq \alpha \leq 1 \text{ and } r \geq 1. \quad (1.3)$$

Taking $r = 1$ in the inequality (1.3), we get

$$\omega^2(T) \leq \frac{1}{2} \| |T|^2 + \frac{1}{4} \| |T|^{4\alpha} + |T^*|^{4(1-\alpha)} \| \quad \text{for all } 0 \leq \alpha \leq 1. \quad (1.4)$$

In [22], the authors showed that if $T \in \mathbb{B}(\mathcal{H})$ has the Cartesian decomposition $T = X + iY$, then

$$\omega(T) \leq \frac{1}{2} \| |X|^{2\alpha} + |X|^{2(1-\alpha)} + |Y|^{2\alpha} + |Y|^{2(1-\alpha)} \| \quad \text{for all } 0 \leq \alpha \leq 1. \quad (1.5)$$

EL-Haddad and Kittaneh in [14] showed that if $S, T \in \mathbb{B}(\mathcal{H})$, $0 \leq \alpha \leq 1$ and $r \geq 1$, then

$$\|S + T\|^r \leq 2^{r-2} \left(\| |S|^{2\alpha r} + |T|^{2\alpha r} \| + \| |S^*|^{2(1-\alpha)r} + |T^*|^{2(1-\alpha)r} \| \right).$$

Also, they proved the following numerical radius and norm inequality for the sum of operators

$$\omega^r(S + T) \leq 2^{r-2} \left\| |S|^{2\alpha r} + |T|^{2\alpha r} + |S^*|^{2(1-\alpha)r} + |T^*|^{2(1-\alpha)r} \right\| \quad (1.6)$$

for $S, T \in \mathbb{B}(\mathcal{H})$, $0 \leq \alpha \leq 1$ and $r \geq 1$.

For more similar inequalities, we refer to [3, 4, 1, 8, 9, 17, 19, 21, 24, 27], as well as their respective references.

In the following, we focus on the semi-Hilbert spaces. Any positive operator $A \in \mathbb{B}(\mathcal{H})$ defines a positive semidefinite sesquilinear form:

$$\begin{aligned} \langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} &\longrightarrow \mathbb{C}, \\ \langle x, y \rangle_A &= \langle Ax, y \rangle \quad (x, y \in \mathcal{H}). \end{aligned}$$

Note that, for any $A \in \mathbb{B}(\mathcal{H})$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are, respectively, the rang space of A and the null space of A . We denote by $\| \cdot \|_A$ the semi-norm induced by $\langle \cdot, \cdot \rangle_A$, that is, $\|x\|_A = \sqrt{\langle x, x \rangle_A}$ for all $x \in \mathcal{H}$. Observe that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. The semi-normed space $(\mathcal{H}, \| \cdot \|_A)$ is a complete space if and only if every Cauchy sequence is convergent to a member of it or, in other words, $\mathcal{R}(A)$ is closed in $\mathcal{H}$. For $T \in \mathbb{B}(\mathcal{H})$, the A -operator semi-norm of T is defined by

$$\|T\|_A = \sup\{\|Tx\|_A : x \in \overline{\mathcal{R}(A)}, \|x\|_A = 1\} < +\infty.$$

An element $Z \in \mathbb{B}(\mathcal{H})$ is called an A -adjoint operator of $T \in \mathbb{B}(\mathcal{H})$, if for every $x, y \in \mathcal{H}$, $\langle Tx, y \rangle_A = \langle x, Zy \rangle_A$. In other words, $AZ = T^*A$. It follows from

[13] that T admits an A -adjoint operator if and only if $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A)$, and so there exists a unique operator $Y \in \mathbb{B}(\mathcal{H})$ such that $AY = T^*A$ and $\mathcal{R}(Y) \subseteq \overline{\mathcal{R}(A)}$. The set of all such operators is denoted by $\mathbb{B}_A(\mathcal{H})$. If $S, T \in \mathbb{B}_A(\mathcal{H})$, then $ST \in \mathbb{B}_A(\mathcal{H})$ and $(ST)^{*A} = T^{*A}S^{*A}$. Moreover, if $\mathbb{B}^A(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$ whose A -operator seminorm is finite, then $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}^A(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$. If the operator A is positive onto, then for every operator $T \in \mathbb{B}(\mathcal{H})$, we have $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A) = \mathcal{H}$, and so $\mathbb{B}(\mathcal{H}) \subseteq \mathbb{B}_A(\mathcal{H})$.

An operator $T \in \mathbb{B}_A(\mathcal{H})$ is called A -selfadjoint if AT is selfadjoint, i.e., $AT = T^*A$, or simply $\langle Tx, x \rangle_A \in \mathbb{R}$ for all $x \in \mathcal{H}$. For $T \in \mathbb{B}_A(\mathcal{H})$, $T = T^{*A}$ if and only if T is A -selfadjoint and $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$. Also, T is called A -positive if AT is positive. Note that, if T is A -selfadjoint, then $T \in \mathbb{B}_A(\mathcal{H})$. An operator $T \in \mathbb{B}_A(\mathcal{H})$ is said to be A -normal if $TT^{*A} = T^{*A}T$. The fact that every selfadjoint operator is normal does not hold in this case, i.e., an A -selfadjoint operators not necessarily A -normal, see [7, Example 5.1]. Indeed, this property holds if T commutes with A . Note that if T is A -positive, then

$$\|T\|_A = \sup\{|\langle Tx, x \rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}.$$

It follows from for every $T \in \mathbb{B}_A(\mathcal{H})$, the operators $T^{*A}T$ and TT^{*A} are both A -positive, that

$$\|T^{*A}T\|_A = \|TT^{*A}\|_A = \|T\|_A^2 = \|T^{*A}\|_A^2.$$

For more investigations about semi-Hilbert spaces and their details, we refer to [5, 6, 29] and references therein.

The definition of the A -numerical radius of an operator $T \in \mathbb{B}_A(\mathcal{H})$ reads as follows:

$$\begin{aligned} w_{\mathbf{A}}(T) &:= \sup\{|\langle Tx, x \rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\} \\ &= \sup\{|\langle ATx, x \rangle| : x \in \mathcal{H}, \|x\|_A = 1\}. \end{aligned}$$

It follows that for any $T \in \mathbb{B}_A(\mathcal{H})$, that $w_{\mathbf{A}}(T^{*A}) = w_{\mathbf{A}}(T)$.

The A -numerical radius and the numerical radius of operators in Hilbert spaces have been extensively studied over the past few decades. They are valuable for understanding the role of operators in various applications, including numerical analysis, physics, information theory, differential equations, and the stability theory of difference approximations for hyperbolic initial value problems (see references [5, 6, 16, 29]). The A -numerical radius is often used as a more reliable indicator of the convergence rate of iterative methods compared to the spectral radius. It's important to note that computing the A -numerical radius is an optimization problem. For more information about semi-Hilbert spaces, we refer readers to references [7, 11, 10, 12, 20, 23, 25] and the associated literature.

Our findings will utilize the angle $\angle_{x,y}$ between two vectors $x, y \in \mathbb{C}^n$ or $x, y \in \mathcal{H}$. For such x and y , the Cauchy-Schwarz inequality states that $|\langle x, y \rangle| \leq \|x\| \|y\|$. From this, the angle between the non-zero vectors x and y can be defined by

$$\angle_{x,y} = \cos^{-1} \left(\frac{|\langle x, y \rangle|}{\|x\| \|y\|} \right).$$

Sababheh et al. [26, Corollary 2.1] proved a refinement of the Cauchy-Schwarz inequality as follows

$$|\langle x, y \rangle| \leq \mu(\theta_{x,y}) \|x\| \|y\|, \quad (1.7)$$

where $x, y \in \mathcal{H}$, $\theta_{x,y} = \angle_{x,y}$ and $\mu(\psi) := \frac{1}{4}(2 + \cos \psi \cot \psi \log(\frac{1+\sin \psi}{1-\sin \psi}))$.

The function μ is defined on $\mathbb{R} - \{n\pi\}_{n \in \mathbb{Z}}$. Since $\lim_{\psi \rightarrow n\pi} \mu(\psi) = 1$, we define $\mu(n\pi) := 1$. It was also shown that the function μ is decreasing on the interval $[0, \frac{\pi}{2}]$ and is increasing in $[\frac{\pi}{2}, \pi]$ and the inequalities $\frac{1}{2} \leq \mu(\psi) \leq 1$ hold for all $\psi \geq 0$. In fact, the following inequalities hold

(i) If $0 \leq \theta_1 < \theta_2 \leq \frac{\pi}{2}$, then

$$\mu(\theta_2) \leq \mu(\theta_1); \quad (1.8)$$

(ii) If $\frac{\pi}{2} \leq \theta_1 < \theta_2 \leq \pi$, then

$$\mu(\theta_1) \leq \mu(\theta_2). \quad (1.9)$$

Moreover, they derived a reverse version of the Cauchy-Schwarz inequality as follows: [26, Remark 2.3]:

$$\gamma_t(\theta_{x,y}) \|x\| \|y\| \leq |\langle x, y \rangle|, \quad (1.10)$$

where

$x, y \in \mathcal{H}$, $\theta_{x,y} = \angle_{x,y}$, $t \in [0, 1]$, $\gamma_t(\psi) := 1 - \frac{1}{2r_t}(1 - \sqrt{\cos^2 \psi + (2t-1)^2 \sin^2 \psi})$ and $r_t = \min\{t, 1-t\}$. $\gamma_t(\psi) := 1 - \frac{1}{2r_t}(1 - \sqrt{\cos^2 \psi + (2t-1)^2 \sin^2 \psi})$ is periodic in ψ with period π and $\gamma_t(\psi) = \gamma_{1-t}(\psi)$ for $t \in [0, 1]$. Furthermore, the function γ is decreasing on the interval $[0, \frac{\pi}{2}]$ and is increasing in $[\frac{\pi}{2}, \pi]$ and the inequalities $0 \leq \gamma_t(\psi) \leq 1$ holds, and also $\max_{0 \leq t \leq 1} \gamma_t(\psi) = \cos(\psi)$.

This paper aims to explore generalized inequalities involving the A -numerical radius within the framework of semi-Hilbert spaces. Especially, we generalize the inequalities (1.2), (1.4) and (1.5). By examining these inequalities, we can gain deeper insights into the spectral properties and stability of operators in these generalized spaces. The results obtained could have significant ramifications for both pure mathematics and applied fields, where such operators are frequently encountered.

2. Some bounds for A -numerical radius. In this section, we present upper bounds related to the A -numerical radius within the context of semi-Hilbert spaces. Our findings offer a thorough analysis of its properties and implications. Before we delve into the main theorems and results, we will state some relevant lemmas related to this topic.

The A -Cauchy-Schwarz inequality is stated as follows:

$$|\langle x, y \rangle_A|^2 \leq \|x\|_A \|y\|_A. \quad (2.1)$$

where $x, y \in \mathcal{H}$; see [29]. In the following lemma, we present a reversed and refined version of the A -Cauchy-Schwarz inequality.

LEMMA 2.1. *Let $\mathcal{H}$ be a Hilbert space, $A \in \mathbb{B}(\mathcal{H})$ be a positive operator and $x, y \in \mathcal{H}$. Then*

$$\gamma_t(\theta_{x,y}) \|x\|_A \|y\|_A \leq |\langle x, y \rangle_A| \leq \mu(\theta_{x,y}) \|x\|_A \|y\|_A, \quad (2.2)$$

where $\theta_{x,y} = \angle_{A^{\frac{1}{2}}x, A^{\frac{1}{2}}y}$, $t \in [0, 1]$, $r_t = \min\{t, 1-t\}$, $\gamma_t(\psi) := 1 - \frac{1}{2r_t}(1 - \sqrt{\cos^2 \psi + (2t-1)^2 \sin^2 \psi})$ and $\mu(\psi) := \frac{1}{4}(2 + \cos \psi \cot \psi \log(\frac{1+\sin \psi}{1-\sin \psi}))$.

Proof. Applying the inequality (1.7), we have

$$\gamma_t^2(\theta_{x,y}) \|x\|_A^2 \|y\|_A^2 = \gamma_t^2(\theta_{x,y}) \|A^{\frac{1}{2}}x\|^2 \|A^{\frac{1}{2}}y\|^2 \leq |\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y \rangle|^2 = |\langle x, y \rangle_A|^2.$$

where $\theta_{x,y} = \angle_{A^{\frac{1}{2}}x, A^{\frac{1}{2}}y}$, $t \in [0, 1]$, $\gamma_t(\psi) := 1 - \frac{1}{2r_t}(1 - \sqrt{\cos^2 \psi + (2t-1)^2 \sin^2 \psi})$ and $r_t = \min\{t, 1-t\}$. Moreover, using the inequality (1.10), we have

$$\begin{aligned} |\langle x, y \rangle_A|^2 &= |\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y \rangle_A|^2 \\ &\leq \mu^2(\theta_{x,y}) \|A^{\frac{1}{2}}x\|^2 \|A^{\frac{1}{2}}y\|^2, \\ &= \mu^2(\theta_{x,y}) \|x\|_A^2 \|y\|_A^2, \end{aligned}$$

where $\theta_{x,y} = \angle_{A^{\frac{1}{2}}x, A^{\frac{1}{2}}y}$ and $\mu(\psi) := \frac{1}{4}(2 + \cos \psi \cot \psi \log(\frac{1+\sin \psi}{1-\sin \psi}))$. □

LEMMA 2.2. Let $A, T \in \mathbb{B}(\mathcal{H})$ be such that A is a positive onto operator and T have the polar decomposition $T = U|T|$, $x, y \in \mathcal{H}$ and f, g be nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, +\infty)$). Then

$$|\langle Tx, y \rangle_A|^2 \leq \mu^2(\theta_{T,x,y}) \langle (f(|T|))^* A f(|T|) x, x \rangle_A \langle (Ug(|T|)(g(|T|))^* U^* A) y, y \rangle_A,$$

where $\theta_{T,x,y} = \angle_{f(|T|)x, g(|T|)^* A U^* A y}$. In particular, for $0 \leq \alpha \leq 1$

$$|\langle Tx, y \rangle_A|^2 \leq \mu^2(\theta_{T,x,y}) \langle |T|^{\alpha* A} |T|^\alpha x, x \rangle_A \left\langle \left(U|T|^{1-\alpha} |T|^{1-\alpha* A} U^* A \right) y, y \right\rangle_A,$$

where $\theta_{T,x,y} = \angle_{|T|^\alpha x, |T|^{1-\alpha* A} U^* A y}$.

Proof. First, note that, it follows from $\mathcal{R}(A) = \mathcal{H}$, that $\mathcal{R}(T^* A), \mathcal{R}(|T|A), \mathcal{R}(U^* A), \mathcal{R}(UA) \subseteq \mathcal{R}(A)$, and so $T, |T|, U, U^* \in \mathbb{B}_A(\mathcal{H})$. If $x, y \in \mathcal{H}$, then we have

$$\begin{aligned} |\langle Tx, y \rangle_A|^2 &= |\langle U|T|x, y \rangle_A|^2 \\ &= |\langle Ug(|T|)f(|T|)x, y \rangle_A|^2 \\ &= |\langle f(|T|)x, (g(|T|))^* A U^* A y \rangle_A|^2 \\ &\leq \mu^2(\theta_{T,x,y}) \|f(|T|)x\|_A^2 \|(g(|T|))^* A U^* A y\|_A^2 \\ &= \mu^2(\theta_{T,x,y}) \langle f(|T|)x, f(|T|)x \rangle_A \langle (g(|T|))^* A U^* A y, g(|T|)^* A U^* A y \rangle_A \\ &= \mu^2(\theta_{T,x,y}) \langle (f(|T|))^* A f(|T|)x, x \rangle_A \langle [g(|T|)^* A U^* A]^* A g(|T|)^* A U^* A y, y \rangle \\ &= \mu^2(\theta_{T,x,y}) \langle (f(|T|))^* A f(|T|)x, x \rangle_A \langle [AUg(|T|)] g(|T|)^* A U^* A y, y \rangle \\ &\quad (\text{since } (ST)^* A = T^* A S^* A \text{ and } AS^* A = S^* A) \\ &= \mu^2(\theta_{T,x,y}) \langle (f(|T|))^* A f(|T|)x, x \rangle_A \langle (Ug(|T|)(g(|T|))^* A U^* A) y, y \rangle_A, \end{aligned}$$

where $\theta_{T,x,y} = \angle_{f(|T|)x, g(|T|)^* A U^* A y}$. For the specific case, it suffices to use $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, where $0 \leq \alpha \leq 1$. □

THEOREM 2.3. *Let $A, T \in \mathbb{B}(\mathcal{H})$ be such that A is a positive onto operator and T have the polar decomposition $T = U|T|$, $x \in \mathcal{H}$, $\theta_{T,x} = \angle_{f(|T|)x, g(|T|)^*AU^*Ax}$ and f, g be nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, +\infty)$). Then*

$$w_A(T) \leq \frac{\mu(\theta)}{2} \|(f(|T|))^*A f(|T|) + Ug(|T|)(g(|T|))^*AU^*A\|_A,$$

either $0 \leq \theta < \theta_{T,x} \leq \frac{\pi}{2}$ for all unit vectors $x \in \mathcal{H}$, or $\frac{\pi}{2} \leq \theta_{T,x} < \theta \leq \pi$ for all A -unit vectors $x \in \mathcal{H}$.

Proof. Assume $x \in \mathcal{H}$. Using Lemma 2.2, we have

$$\begin{aligned} |\langle Tx, x \rangle_A| &\leq \mu(\theta_{T,x}) \sqrt{\langle (f(|T|))^*A f(|T|)x, x \rangle_A} \sqrt{\langle (Ug(|T|)(g(|T|))^*AU^*A)x, x \rangle_A} \\ &\leq \frac{\mu(\theta_{T,x})}{2} (\langle (f(|T|))^*A f(|T|) + (Ug(|T|)(g(|T|))^*AU^*A)x, x \rangle_A) \\ &\quad \text{(by the arithmetic-geometric mean inequality)} \\ &\leq \frac{\mu(\theta_{T,x})}{2} \|(f(|T|))^*A f(|T|) + Ug(|T|)(g(|T|))^*AU^*A\|_A \|x\|_A, \end{aligned}$$

where $\theta_{T,x} = \angle_{f(|T|)x, g(|T|)^*AU^*Ax}$. If either $0 \leq \theta < \theta_{T,x} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{T,x} < \theta \leq \pi$ for all A -unit vectors $x \in \mathcal{H}$, then by (1.8) and (1.9), we have $\mu(\theta_{T,x}) \leq \mu(\theta)$. Therefore,

$$\begin{aligned} |\langle Tx, x \rangle_A| &\leq \frac{\mu(\theta_{T,x})}{2} \|(f(|T|))^*A f(|T|) + Ug(|T|)(g(|T|))^*AU^*A\|_A \\ &\leq \frac{\mu(\theta)}{2} \|(f(|T|))^*A f(|T|) + Ug(|T|)(g(|T|))^*AU^*A\|_A. \end{aligned}$$

By taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequalities, we get the desired results. $\square$

As a special case of Theorem 2.3 for $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ ($0 \leq \alpha \leq 1$), we obtain the following result.

COROLLARY 2.4. *Let $A, T \in \mathbb{B}(\mathcal{H})$ be such that A is a positive onto operator and T have the polar decomposition $T = U|T|$, $x \in \mathcal{H}$, $0 \leq \alpha \leq 1$ and $\theta_{T,x} = \angle_{|T|^\alpha x, |T|^{1-\alpha}AU^*Ax}$. Then*

$$w_A(T) \leq \frac{\mu(\theta)}{2} \left\| |T|^{\alpha}AU^*A|T|^\alpha + U|T|^{1-\alpha}|T|^{1-\alpha}AU^*A \right\|_A,$$

where either $0 \leq \theta < \theta_{T,x} \leq \frac{\pi}{2}$ for all unit vectors $x \in \mathcal{H}$, or $\frac{\pi}{2} \leq \theta_{T,x} < \theta \leq \pi$ for all A -unit vectors $x \in \mathcal{H}$.

EXAMPLE 2.5. If we consider the matrix $T = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, the positive semidefinite matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ and the A -unit vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then by the polar decomposition

$$T = U|T| = \begin{bmatrix} 0.7673 & 0.6392 \\ -0.6425 & 0.7681 \end{bmatrix} \begin{bmatrix} 2.0523 & -0.9032 \\ -0.9032 & 5.7582 \end{bmatrix},$$

we have

$$\begin{aligned} 0.428789 &\lesssim \cos(\theta_{T,x}) \\ &= \frac{\langle |T|^{\frac{1}{2}}x, |T|^{\frac{1}{2}}{}^*AU^*Ax \rangle}{\| |T|^{\frac{1}{2}}x \| \| |T|^{\frac{1}{2}}{}^*AU^*Ax \|} \\ &= \frac{(1.413x_1 - 0.238x_2)(0.932x_1 - 1.091x_2) + (-0.238x_1 + 2.388x_2)(1.344x_1 + 1.987x_2)}{\sqrt{(1.413x_1 - 0.238x_2)^2 + (-0.238x_1 + 2.388x_2)^2} \sqrt{(0.932x_1 - 1.091x_2)^2 + (1.362x_1 + 1.987x_2)^2}} \lesssim 0.681583, \end{aligned}$$

whence

$$47.032532^\circ \lesssim \theta_{T,x} \lesssim 64.609839^\circ.$$

Now, if we take $\alpha = \frac{1}{2}$ in Corollary 2.4, we obtain

$$w_A(T) \simeq 7.1589 \preceq \frac{\mu(\theta)}{2} \left\| |T|^{\frac{1}{2}}{}^*A|T|^{\frac{1}{2}} + U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}{}^*AU^*A \right\|_A \simeq \frac{\mu(\theta)}{2} 31.1658,$$

where $0 \leq \theta < 47.032532$.

REMARK 2.6. If we take $A = I$ in Corollary 2.4, we will have:

$$\begin{aligned} w(T) &\leq \frac{\mu(\theta)}{2} \left\| |T|^{\alpha}{}^*|T|^{\alpha} + U|T|^{1-\alpha}|T|^{1-\alpha}{}^*U^* \right\| \\ &= \frac{\mu(\theta)}{2} \left\| |T|^{2\alpha} + U|T|^{2(1-\alpha)}U^* \right\| \\ &= \frac{\mu(\theta)}{2} \left\| |T|^{2\alpha} + |T^*|^{2(1-\alpha)} \right\|, \end{aligned}$$

where $0 \leq \alpha \leq 1$ and either $0 \leq \theta < \theta_{T,x} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{T,x} < \theta \leq \pi$ for all unit vectors $x \in \mathcal{H}$. For $\alpha = \frac{1}{2}$, we have

$$\begin{aligned} w(T) &\leq \frac{\mu(\theta)}{2} \| |T| + |T^*| \| \\ &\leq \frac{1}{2} \| |T| + |T^*| \|. \end{aligned}$$

These inequalities indicate that Corollary 2.4 generalizes and refines the inequality (1.2) for the case when the power is one.

In the theorem presented below, we generalize and refine the inequality given in the inequality (1.5).

THEOREM 2.7. *Let $A, T \in \mathbb{B}(\mathcal{H})$ be such that A is a positive onto operator and T have the polar decomposition $T = U|T|$, $\mu(\psi) = \frac{1}{4}(2 + \cos \psi \cot \psi \log(\frac{1+\sin \psi}{1-\sin \psi}))$, and let f, g be nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, +\infty)$). If either $0 \leq \theta < \theta_{T,x} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{T,x} < \theta \leq \pi$ for all A -unit vectors $x \in \mathcal{H}$, then*

$$\omega_A(T) \leq \alpha \|T\|_A + \frac{(1-\alpha)}{2} \mu(\theta) \|(f(|T|))^*{}^*A f(|T|) + U g(|T|)(g(|T|))^*{}^*AU^*A\|_A,$$

where $\theta_{T,x} = \angle_{f(|T|)x, (g(|T|))^*{}^*AU^*Ax}$ and $0 \leq \alpha \leq 1$.

Proof. Assume $T \in \mathbb{B}(\mathcal{H})$ has the polar decomposition $T = U|T|$ and $x \in \mathcal{H}$. Then

$$\begin{aligned}
 & |\langle Tx, y \rangle|_A \\
 &= \alpha |\langle Tx, x \rangle|_A + (1 - \alpha) |\langle Tx, x \rangle|_A \\
 &\leq \alpha \|Tx\|_A \|x\|_A + (1 - \alpha) |\langle Tx, x \rangle|_A \\
 &\leq \alpha \|Tx\|_A \|x\|_A \\
 &\quad + (1 - \alpha) \mu(\theta_{T,x}) \sqrt{\langle (f(|T|))^* f(|T|) x, x \rangle_A \langle (Ug(|T|)(g(|T|))^* U^* x, x \rangle_A} \\
 &\hspace{25em} \text{(by Lemma (2.2))} \\
 &\leq \alpha \|Tx\|_A \|x\|_A \\
 &\quad + \frac{(1 - \alpha)}{2} \mu(\theta_{T,x}) (\langle (f(|T|))^* f(|T|) x, x \rangle_A + \langle (Ug(|T|)(g(|T|))^* U^* x, x \rangle_A) \\
 &\hspace{25em} \text{(by the arithmetic-geometric mean inequality)} \\
 &= \alpha \|Tx\|_A \|x\|_A + \frac{(1 - \alpha)}{2} \mu(\theta_{T,x}) \langle ((f(|T|))^* f(|T|) + Ug(|T|)(g(|T|))^* U^*) x, x \rangle_A \\
 &\leq \alpha \|T\|_A \|x\|_A^2 + \frac{(1 - \alpha)}{2} \mu(\theta_{T,x}) \|(f(|T|))^* f(|T|) + Ug(|T|)(g(|T|))^* U^*\|_A \|x\|_A,
 \end{aligned}$$

where $\theta_{T,x} = \angle_{f(|T|)x, g(|T|)^* U^* A x}$ and $0 \leq \alpha \leq 1$. Hence,

$$\begin{aligned}
 |\langle Tx, x \rangle| &\leq \alpha \|T\|_A \|x\|_A^2 \\
 &\quad + \frac{(1 - \alpha)}{2} \mu(\theta_{T,x}) \|(f(|T|))^* f(|T|) + Ug(|T|)(g(|T|))^* U^*\|_A \|x\|_A \quad (2.3)
 \end{aligned}$$

for all $0 \leq \alpha \leq 1$. Using the inequality (2.3), we have

(i) If $0 \leq \theta < \theta_{T,x} \leq \frac{\pi}{2}$ for all A -unit vectors $x \in \mathcal{H}$, then

$$\begin{aligned}
 |\langle Tx, x \rangle| &\leq \alpha \|T\|_A \|x\|_A^2 \\
 &\quad + \frac{(1 - \alpha)}{2} \mu(\theta_{T,x}) \|(f(|T|))^* f(|T|) + Ug(|T|)(g(|T|))^* U^*\|_A \|x\|_A \\
 &\leq \alpha \|T\|_A \|x\|_A^2 \\
 &\quad + \frac{(1 - \alpha)}{2} \mu(\theta) \|(f(|T|))^* f(|T|) + Ug(|T|)(g(|T|))^* U^*\|_A \|x\|_A.
 \end{aligned}$$

By taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get

$$\omega(T) \leq \alpha \|T\|_A + \frac{(1 - \alpha)}{2} \mu(\theta) \|(f(|T|))^* f(|T|) + Ug(|T|)(g(|T|))^* U^*\|_A.$$

(ii) If $\frac{\pi}{2} \leq \theta_{T,x} < \theta \leq \pi$ for all A -unit vectors $x \in \mathcal{H}$, we have

$$\begin{aligned}
 |\langle Tx, x \rangle| &\leq \alpha \|T\|_A \|x\|_A^2 \\
 &\quad + \frac{(1 - \alpha)}{2} \mu(\theta_{T,x}) \|(f(|T|))^* f(|T|) + Ug(|T|)(g(|T|))^* U^*\|_A \|x\|_A \\
 &\leq \alpha \|T\|_A \|x\|_A^2 \\
 &\quad + \frac{(1 - \alpha)}{2} \mu(\theta) \|(f(|T|))^* f(|T|) + Ug(|T|)(g(|T|))^* U^*\|_A \|x\|_A.
 \end{aligned}$$

Now, by taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$, we deduce the required result. $\square$

REMARK 2.8. Consider the functions $f(t) = t^p$ and $g(t) = t^{1-p}$ ($0 \leq p \leq 1$), in Theorem 2.7, then we have

$$\omega_A(T) \leq \alpha \|T\|_A + \frac{(1-\alpha)}{2} \mu(\theta) \left\| (|T|^p)^{*A} |T|^p + U |T|^{1-p} (|T|^{1-p})^{*A} U^{*A} \right\|_A,$$

where $\theta_{T,x} = \angle_{|T|^p x, (|T|^{1-p})^{*A} U^{*A} x}$, either $0 \leq \theta < \theta_{T,x} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{T,x} < \theta \leq \pi$ for all A -unit vectors $x \in \mathcal{H}$ and $0 \leq \alpha \leq 1$. For an special case $A = I$, we abtian

$$\omega(T) \leq \alpha \|T\| + \frac{(1-\alpha)}{2} \mu(\theta) \left\| |T|^{2p} + |T^*|^{2(1-p)} \right\| \quad \text{for } 0 \leq \alpha \leq 1.$$

This generalizes the inequality (1.4) to the case, where the power is one.

THEOREM 2.9. Let $A, T \in \mathbb{B}(\mathcal{H})$ be such that A is a positive onto operator and T have the Cartesian decomposition $T = X + iY$, $x, y \in \mathcal{H}$ be A -unit vectors, $X = U|X|$, $Y = V|Y|$ be the polar decompositions of X and Y , and let f, g be nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, +\infty)$), $\theta_{X,x,y} = \angle_{f(|X|)x, g(|X|)^{*A} U^{*A} y}$, $\theta_{Y,x,y} = \angle_{f(|Y|)x, g(|Y|)^{*A} V^{*A} y}$, and $\mu(\psi) = \frac{1}{4}(2 + \cos \psi \cot \psi \log(\frac{1+\sin \psi}{1-\sin \psi}))$. Then

- (i) If $0 \leq \theta' < \theta_{X,x,y} \leq \frac{\pi}{2}$ and $0 \leq \theta'' < \theta_{Y,x,y} \leq \frac{\pi}{2}$ for all A -unit vectors $x, y \in \mathcal{H}$, then

$$\begin{aligned} \|T\|_A^2 &\leq \mu^2(\theta) \left\| (f(|X|))^{*A} f(|X|) + (f(|Y|))^{*A} f(|Y|) \right\|_A \\ &\quad \left\| U g(|X|) (g(|X|))^{*A} U^{*A} + U g(|Y|) (g(|Y|))^{*A} U^{*A} \right\|_A, \end{aligned}$$

where $\theta = \min\{\theta', \theta''\}$;

- (ii) If $\frac{\pi}{2} \leq \theta_{X,x,y} < \theta' \leq \pi$ and $\frac{\pi}{2} \leq \theta_{Y,x,y} < \theta'' \leq \pi$ for all A -unit vectors $x, y \in \mathcal{H}$, then

$$\begin{aligned} \|T\|_A^2 &\leq \mu^2(\theta) \left\| (f(|X|))^{*A} f(|X|) + (f(|Y|))^{*A} f(|Y|) \right\|_A \\ &\quad \left\| U g(|X|) (g(|X|))^{*A} U^{*A} + U g(|Y|) (g(|Y|))^{*A} U^{*A} \right\|_A, \end{aligned}$$

where $\theta = \max\{\theta', \theta''\}$.

Proof. Assume $T \in \mathbb{B}(\mathcal{H})$ has the Cartesian decomposition $T = X + iY$, $x, y \in \mathcal{H}$ are A -unit vectors, $\theta_{X,x,y} = \angle_{f(|X|)x, g(|X|)^{*A} U^{*A} y}$ and $\theta_{Y,x,y} = \angle_{f(|Y|)x, g(|Y|)^{*A} V^{*A} y}$.

We have

$$\begin{aligned}
& |\langle Tx, y \rangle|_A \\
&= |\langle (X + iY)x, y \rangle|_A \\
&\leq |\langle Xx, y \rangle|_A + |\langle Yx, y \rangle|_A \quad (\text{by the triangle inequality}) \\
&\leq \mu(\theta_{X,x,y}) \sqrt{\langle (f(|X|))^* f(|X|)x, x \rangle_A \langle (Ug(|X|)(g(|X|))^* U^* y, y \rangle_A} \\
&\quad + \mu(\theta_{Y,x,y}) \sqrt{\langle (f(|Y|))^* f(|Y|)x, x \rangle_A \langle (Ug(|Y|)(g(|Y|))^* U^* y, y \rangle_A} \\
&\quad (\text{by Lemma 2.2}) \\
&\leq \mu_\theta \left(\sqrt{\langle (f(|X|))^* f(|X|)x, x \rangle_A \langle (Ug(|X|)(g(|X|))^* U^* y, y \rangle_A} \right. \\
&\quad \left. + \sqrt{\langle (f(|Y|))^* f(|Y|)x, x \rangle_A \langle (Ug(|Y|)(g(|Y|))^* U^* y, y \rangle_A} \right) \\
&\quad (\text{by } \mu_\theta = \max\{\mu(\theta_{X,x,y}), \mu(\theta_{Y,x,y})\}) \\
&\leq \mu_\theta \times \sqrt{\langle (f(|X|))^* f(|X|)x, x \rangle_A + \langle (f(|Y|))^* f(|Y|)x, x \rangle_A} \\
&\quad \sqrt{\langle (Ug(|X|)(g(|X|))^* U^* y, y \rangle_A + \langle (Ug(|Y|)(g(|Y|))^* U^* y, y \rangle_A} \\
&\quad (\text{by the Cauchy-Schwarz inequality}),
\end{aligned}$$

whence

$$\begin{aligned}
& |\langle Tx, y \rangle|_A \\
&\leq \mu_\theta \times \sqrt{\langle (f(|X|))^* f(|X|)x, x \rangle_A + \langle (f(|Y|))^* f(|Y|)x, x \rangle_A} \\
&\quad \sqrt{\langle [Ug(|X|)(g(|X|))^* U^*]^* y, y \rangle_A + \langle (Ug(|Y|)(g(|Y|))^* U^* y, y \rangle_A}, \quad (2.4)
\end{aligned}$$

where $\mu_\theta = \max\{\mu(\theta_{S,x,y}), \mu(\theta_{T,x,y})\}$. Hence,

$$\begin{aligned}
& |\langle Tx, y \rangle|_A \\
&\leq \mu_\theta \times \sqrt{\langle (f(|X|))^* f(|X|)x, x \rangle_A + \langle (f(|Y|))^* f(|Y|)x, x \rangle_A} \\
&\quad \sqrt{\langle (Ug(|X|)(g(|X|))^* U^* y, y \rangle_A + \langle (Ug(|Y|)(g(|Y|))^* U^* y, y \rangle_A} \\
&\leq \mu_\theta \times \sqrt{\langle ((f(|X|))^* f(|X|) + (f(|Y|))^* f(|Y|))x, x \rangle_A} \\
&\quad \sqrt{\langle (Ug(|X|)(g(|X|))^* U^* + (Ug(|Y|)(g(|Y|))^* U^*) y, y \rangle_A} \\
&\leq \mu_\theta \|((f(|X|))^* f(|X|) + (f(|Y|))^* f(|Y|))\|_A^{\frac{1}{2}} \\
&\quad \|(Ug(|X|)(g(|X|))^* U^* + (Ug(|Y|)(g(|Y|))^* U^*)\|_A^{\frac{1}{2}}, \quad (2.5)
\end{aligned}$$

where $\mu_\theta = \max\{\mu(\theta_{X,x,y}), \mu(\theta_{Y,x,y})\}$. Hence, we have two cases as follow:

(i) If $0 \leq \theta' < \theta_{X,x,y} \leq \frac{\pi}{2}$ and $0 \leq \theta'' < \theta_{Y,x,y} \leq \frac{\pi}{2}$ for all A -unit vectors $x, y \in \mathcal{H}$,

then put $\theta = \min\{\theta', \theta''\}$. It follows from the monotonicity of μ (1.8), $\theta \leq \theta'$ and $\theta \leq \theta''$ that

$$\mu(\theta') \leq \mu(\theta) \quad \text{and} \quad \mu(\theta'') \leq \mu(\theta). \quad (2.6)$$

Moreover, by the monotonicity of μ (1.8), $\theta' < \theta_{X,x,y}$ and $\theta'' < \theta_{Y,x,y}$, we have

$$\mu(\theta_{X,x,y}) \leq \mu(\theta') \quad \text{and} \quad \mu(\theta_{Y,x,y}) \leq \mu(\theta''). \quad (2.7)$$

Therefore, applying (2.6) and (2.7), we obtain

$$\mu_\theta := \max\{\mu(\theta_{X,x,y}), \mu(\theta_{Y,x,y})\} \leq \max\{\mu(\theta'), \mu(\theta'')\} \leq \mu(\theta). \quad (2.8)$$

Hence, using (2.5) and (2.8), we have

$$\begin{aligned} |\langle Tx, y \rangle| &= |\langle (X + iY)x, y \rangle| \\ &\leq \mu_\theta \|(f(|X|))^* f(|X|) + (f(|Y|))^* f(|Y|)\|_A^{\frac{1}{2}} \\ &\quad \|Ug(|X|)(g(|X|))^* U^{*A} + Ug(|Y|)(g(|Y|))^* U^{*A}\|_A^{\frac{1}{2}}, \\ &\leq \mu(\theta) \|(f(|X|))^* f(|X|) + (f(|Y|))^* f(|Y|)\|_A^{\frac{1}{2}} \\ &\quad \|Ug(|X|)(g(|X|))^* U^{*A} + Ug(|Y|)(g(|Y|))^* U^{*A}\|_A^{\frac{1}{2}}, \end{aligned}$$

where $\theta = \min\{\theta', \theta''\}$. By taking the supremum on the above inequalities overall A -unit vectors $x, y \in \mathcal{H}$, we get the first case.

(ii) If $\frac{\pi}{2} \leq \theta_{X,x,y} < \theta' \leq \pi$ and $\frac{\pi}{2} \leq \theta_{Y,x,y} < \theta'' \leq \pi$ for all A -unit vectors $x, y \in \mathcal{H}$, then put $\theta = \max\{\theta', \theta''\}$. It follows from the monotonicity of μ (1.9), $\theta' \leq \theta$ and $\theta'' \leq \theta$ that

$$\mu(\theta') \leq \mu(\theta) \quad \text{and} \quad \mu(\theta'') \leq \mu(\theta). \quad (2.9)$$

Moreover, by the monotonicity of μ (1.9), $\theta_{X,x,y} < \theta'$ and $\theta_{Y,x,y} < \theta''$, we get

$$\mu(\theta_{X,x,y}) \leq \mu(\theta') \quad \text{and} \quad \mu(\theta_{Y,x,y}) \leq \mu(\theta''). \quad (2.10)$$

Thus, by using (2.9) and (2.10), we obtain

$$\mu_\theta := \max\{\mu(\theta_{X,x,y}), \mu(\theta_{Y,x,y})\} \leq \max\{\mu(\theta'), \mu(\theta'')\} \leq \mu(\theta). \quad (2.11)$$

Applying (2.5) and (2.11), we have

$$\begin{aligned} |\langle Tx, y \rangle| &= |\langle (X + iY)x, y \rangle| \\ &\leq \mu_\theta \|(f(|X|))^* f(|X|) + (f(|Y|))^* f(|Y|)\|_A^{\frac{1}{2}} \\ &\quad \|Ug(|X|)(g(|X|))^* U^{*A} + Ug(|Y|)(g(|Y|))^* U^{*A}\|_A^{\frac{1}{2}}, \\ &\leq \mu(\theta) \|(f(|X|))^* f(|X|) + (f(|Y|))^* f(|Y|)\|_A^{\frac{1}{2}} \\ &\quad \|Ug(|X|)(g(|X|))^* U^{*A} + Ug(|Y|)(g(|Y|))^* U^{*A}\|_A^{\frac{1}{2}}, \end{aligned}$$

By taking the supremum on the above inequalities overall A -unit vectors $x \in \mathcal{H}$, we obtain the second inequality. $\square$

In the special case of Theorem 2.9, when $A = I$, we obtain the following result.

COROLLARY 2.10. *Let $T \in \mathbb{B}(\mathcal{H})$ have the Cartesian decomposition $T = X + iY$, $x, y \in \mathcal{H}$ be unit vectors, $S = U|S|$, $T = V|T|$ be the polar decompositions of X and Y , $\theta_{X,x,y} = \angle_{f(|X|)x,g(|X|)U^*y}$, $\theta_{Y,x,y} = \angle_{f(|Y|)x,g(|Y|)V^*y}$, and let f, g be nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, +\infty)$). Then*

- (i) *If $0 \leq \theta' < \theta_{X,x,y} \leq \frac{\pi}{2}$ and $0 \leq \theta'' < \theta_{Y,x,y} \leq \frac{\pi}{2}$ for all unit vectors $x, y \in \mathcal{H}$, then*

$$\|T\| \leq \mu(\theta) \sqrt{\|f^2(|X|) + f^2(|Y|)\| \|g^2(|X|) + g^2(|Y|)\|},$$

where $\theta = \min\{\theta', \theta''\}$;

- (ii) *If $\frac{\pi}{2} \leq \theta_{X,x,y} < \theta' \leq \pi$ and $\frac{\pi}{2} \leq \theta_{Y,x,y} < \theta'' \leq \pi$ for all unit vectors $x, y \in \mathcal{H}$, then*

$$\|T\| \leq \mu(\theta) \sqrt{\|f^2(|X|) + f^2(|Y|)\| \|g^2(|X|) + g^2(|Y|)\|},$$

where $\theta = \max\{\theta', \theta''\}$.

THEOREM 2.11. *Let $A, T \in \mathbb{B}(\mathcal{H})$ be such that A is a positive onto operator and T have the Cartesian decomposition $T = X + iY$, $x \in \mathcal{H}$, $X = U|X|$, $Y = V|Y|$ be the polar decompositions of X and Y , $\theta_{X,x} = \angle_{f(|X|)x,g(|X|)^*AU^*Ax}$, $\theta_{Y,x} = \angle_{f(|Y|)x,g(|Y|)^*AV^*Ax}$, and let $\mu(\psi) = \frac{1}{4}(2 + \cos \psi \cot \psi \log(\frac{1+\sin \psi}{1-\sin \psi}))$ and f, g be nonnegative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0, +\infty)$). Then*

- (i) *If $0 \leq \theta' < \theta_{X,x} \leq \frac{\pi}{2}$ and $0 \leq \theta'' < \theta_{Y,x} \leq \frac{\pi}{2}$ for all A -unit vectors $x \in \mathcal{H}$, then*

$$\begin{aligned} w_A(T) &\leq \frac{\mu(\theta)}{2} \times \left\| (f(|X|))^*A f(|X|) + (f(|Y|))^*A f(|Y|) \right. \\ &\quad \left. + Ug(|X|)(g(|X|))^*AU^*A + Ug(|Y|)(g(|Y|))^*AU^*A \right\|_A, \end{aligned}$$

where $\theta = \min\{\theta', \theta''\}$;

- (ii) *If $\frac{\pi}{2} \leq \theta_{X,x} < \theta' \leq \pi$ and $\frac{\pi}{2} \leq \theta_{Y,x} < \theta'' \leq \pi$ for all A -unit vectors $x \in \mathcal{H}$, then*

$$\begin{aligned} w_A(T) &\leq \frac{\mu(\theta)}{2} \times \left\| (f(|X|))^*A f(|X|) + (f(|Y|))^*A f(|Y|) \right. \\ &\quad \left. + Ug(|X|)(g(|X|))^*AU^*A + Ug(|Y|)(g(|Y|))^*AU^*A \right\|_A, \end{aligned}$$

where $\theta = \max\{\theta', \theta''\}$.

Proof. Assume $T \in \mathbb{B}(\mathcal{H})$ has the Cartesian decomposition $T = X + iY$, $\theta_{X,x} = \angle_{f(|X|)x, g(|X|)^* A U^* A x}$ and $\theta_{Y,x} = \angle_{f(|Y|)x, g(|Y|)^* A V^* A x}$. Using the inequality (2.4), we have

$$\begin{aligned} & |\langle Tx, x \rangle|_A \\ & \leq \mu_\theta \times \sqrt{\langle (f(|X|))^* A f(|X|)x, x \rangle_A + \langle (f(|Y|))^* A f(|Y|)x, x \rangle_A} \\ & \quad \sqrt{\langle (Ug(|X|)(g(|X|))^* A U^* A)x, x \rangle_A + \langle (Ug(|Y|)(g(|Y|))^* A U^* A)x, x \rangle_A} \\ & \leq \frac{\mu_\theta}{2} \times \langle (f(|X|))^* A f(|X|)x, x \rangle_A + \langle (f(|Y|))^* A f(|Y|)x, x \rangle_A \\ & \quad + \langle (Ug(|X|)(g(|X|))^* A U^* A)x, x \rangle_A + \langle (Ug(|Y|)(g(|Y|))^* A U^* A)x, x \rangle_A \\ & \leq \frac{\mu_\theta}{2} \times \left\| (f(|X|))^* A f(|X|) + (f(|Y|))^* A f(|Y|) \right. \\ & \quad \left. + (Ug(|X|)(g(|X|))^* A U^* A) + (Ug(|Y|)(g(|Y|))^* A U^* A) \right\|_A \|x\|_A, \quad (2.12) \end{aligned}$$

where $\mu_\theta = \max\{\mu(\theta_{X,x}), \mu(\theta_{Y,x})\}$. Hence, we have two cases as follow:

(i) If $0 \leq \theta' < \theta_{X,x} \leq \frac{\pi}{2}$ and $0 \leq \theta'' < \theta_{Y,x} \leq \frac{\pi}{2}$ for all A -unit vectors $x \in \mathcal{H}$, then put $\theta = \min\{\theta', \theta''\}$. It follows from the monotonicity of μ (1.8), $\theta \leq \theta'$ and $\theta \leq \theta''$ that

$$\mu(\theta') \leq \mu(\theta) \quad \text{and} \quad \mu(\theta'') \leq \mu(\theta). \quad (2.13)$$

Moreover, by the monotonicity of μ (1.8), $\theta' < \theta_{X,x}$ and $\theta'' < \theta_{Y,x}$, we have

$$\mu(\theta_{X,x}) \leq \mu(\theta') \quad \text{and} \quad \mu(\theta_{Y,x}) \leq \mu(\theta''). \quad (2.14)$$

Therefore, applying (2.13) and (2.14), we obtain

$$\mu_\theta := \max\{\mu(\theta_{X,x}), \mu(\theta_{Y,x})\} \leq \max\{\mu(\theta'), \mu(\theta'')\} \leq \mu(\theta). \quad (2.15)$$

Applying (2.12) and (2.15), we have

$$\begin{aligned} & |\langle Tx, x \rangle|_A \\ & \leq \frac{\mu_\theta}{2} \times \left\| (f(|X|))^* A f(|X|) + (f(|Y|))^* A f(|Y|) \right. \\ & \quad \left. + Ug(|X|)(g(|X|))^* A U^* A + Ug(|Y|)(g(|Y|))^* A U^* A \right\|_A \|x\|_A \\ & \leq \frac{\mu(\theta)}{2} \times \left\| (f(|X|))^* A f(|X|) + (f(|Y|))^* A f(|Y|) \right. \\ & \quad \left. + Ug(|X|)(g(|X|))^* A U^* A + Ug(|Y|)(g(|Y|))^* A U^* A \right\|_A \|x\|_A, \end{aligned}$$

where $\theta = \min\{\theta', \theta''\}$. By taking the supremum on the above inequalities over an A -unit vector $x \in \mathcal{H}$, we get the first inequality.

(ii) If $\frac{\pi}{2} \leq \theta_{X,x} < \theta' \leq \pi$ and $\frac{\pi}{2} \leq \theta_{Y,x} < \theta'' \leq \pi$ for all A -unit vectors $x \in \mathcal{H}$, then put $\theta = \max\{\theta', \theta''\}$. It follows from the monotonicity of μ (1.9), $\theta' \leq \theta$ and $\theta'' \leq \theta$ that

$$\mu(\theta') \leq \mu(\theta) \quad \text{and} \quad \mu(\theta'') \leq \mu(\theta). \quad (2.16)$$

Moreover, by the monotonicity of μ (1.9), $\theta_{X,x} < \theta'$ and $\theta_{Y,x} < \theta''$, we get

$$\mu(\theta_{S,x}) \leq \mu(\theta') \quad \text{and} \quad \mu(\theta_{T,x}) \leq \mu(\theta''). \quad (2.17)$$

Thus, by using (2.16) and (2.17), we obtain

$$\mu_\theta := \max\{\mu(\theta_{X,x}), \mu(\theta_{Y,x})\} \leq \max\{\mu(\theta'), \mu(\theta'')\} \leq \mu(\theta). \quad (2.18)$$

Applying (2.12) and (2.18), we have

$$\begin{aligned} & |\langle Tx, x \rangle|_A \\ & \leq \frac{\mu_\theta}{2} \times \left\| (f(|X|))^* f(|X|) + (f(|Y|))^* f(|Y|) \right. \\ & \quad \left. + Ug(|X|)(g(|X|))^* U^* + Ug(|Y|)(g(|Y|))^* U^* \right\|_A \|x\|_A \\ & \leq \frac{\mu(\theta)}{2} \times \left\| (f(|X|))^* f(|X|) + (f(|Y|))^* f(|Y|) \right. \\ & \quad \left. + Ug(|X|)(g(|X|))^* U^* + Ug(|Y|)(g(|Y|))^* U^* \right\|_A \|x\|_A, \end{aligned}$$

By taking the supremum of the above inequalities for $x \in \mathcal{H}$ with $\|x\|_A = 1$, we derive the second inequality. $\square$

Next, we will obtain [28, Lemma 2.2] for the semi-Hilbert spaces.

LEMMA 2.12. *Let $\mathcal{H}$ be a Hilbert space, $x, y, z \in \mathcal{H}$ and $p \geq 1$. Then*

$$\langle x, y \rangle_A^{2p} + \langle x, z \rangle_A^{2p} \leq \|x\|_A^{2p} \max\{\|y\|_A^{2p}, \|z\|_A^{2p}\} + \langle y, z \rangle_A^p.$$

Proof. Using [28, Lemma 2.2], we have

$$\langle x, y \rangle^{2p} + \langle x, z \rangle^{2p} \leq \|x\|^{2p} \max\{\|y\|^{2p}, \|z\|^{2p}\} + \langle y, z \rangle^p, \quad (2.19)$$

where $x, y, z \in \mathcal{H}$ and $p \geq 1$. Moreover,

$$\langle x, y \rangle_A^{2p} + \langle x, z \rangle_A^{2p} = \langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y \rangle^{2p} + \langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}z \rangle^{2p}.$$

Applying the recent equation and the inequality (2.19), we have

$$\begin{aligned} \langle x, y \rangle^{2p} + \langle x, z \rangle^{2p} &= \langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y \rangle^{2p} + \langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}z \rangle^{2p} \\ &\leq \|A^{\frac{1}{2}}x\|^{2p} \max\{\|A^{\frac{1}{2}}y\|^{2p}, \|A^{\frac{1}{2}}z\|^{2p}\} + \langle A^{\frac{1}{2}}y, A^{\frac{1}{2}}z \rangle^p \\ &= \|x\|_A^{2p} \max\{\|y\|_A^{2p}, \|z\|_A^{2p}\} + \langle y, z \rangle_A^p, \end{aligned}$$

as required. $\square$

THEOREM 2.13. Let $X, Y \in \mathbb{B}_A(\mathcal{H})$ and $p \geq 1$. Then

$$2^{1-2p}w_A^{2p}(X+Y) \leq w_A^p(XX^{*A} + YY^{*A}) + \|XX^{*A} - YY^{*A}\|_A^p + 2w_A^p(YX^{*A}).$$

Proof. Assume $x \in \mathcal{H}$ is an A -unit vector. We have

$$\begin{aligned} & 2^{-2p}|\langle (X+Y)x, x \rangle_A|^{2p} \\ & \leq 2^{-2p}(|\langle Xx, x \rangle_A| + |\langle Yx, x \rangle_A|)^{2p} \\ & = 2^{-2p}(|\langle x, X^{*A}x \rangle_A| + |\langle x, Y^{*A}x \rangle_A|)^{2p} \\ & \leq |\langle x, X^{*A}x \rangle_A|^{2p} + |\langle x, Y^{*A}x \rangle_A|^{2p} \\ & \quad (\text{by the convexity of } f(t) = t^{2p} \text{ for } p \geq 1) \\ & \leq \max \left\{ \|X^{*A}x\|_A^{2p}, \|Y^{*A}x\|_A^{2p} \right\} + |\langle X^{*A}x, Y^{*A}x \rangle_A|^p \\ & \quad (\text{by Lemma 2.12}) \\ & \leq \frac{1}{2} \left(\|X^{*A}x\|_A^{2p} + \|Y^{*A}x\|_A^{2p} + \left| \|X^{*A}x\|_A^{2p} - \|Y^{*A}x\|_A^{2p} \right| \right) + |\langle X^{*A}x, Y^{*A}x \rangle_A|^p \\ & \quad (\text{since } \max\{u, v\} = \frac{1}{2}(u + v + |u - v|)) \\ & = \frac{1}{2} (\langle XX^{*A} + YY^{*A}x, x \rangle_A^p + |\langle XX^{*A} - YY^{*A}x, x \rangle_A|^p) + |\langle X^{*A}x, Y^{*A}x \rangle_A|^p \\ & \leq \frac{1}{2} (w_A^p(XX^{*A} + YY^{*A}) + \|XX^{*A} - YY^{*A}\|_A^p) + w_A^p(YX^{*A}). \end{aligned}$$

By taking the supremum on the above inequalities overall A -unit vectors $x \in \mathcal{H}$, we get the desired inequality. $\square$

In the next theorem, we show [10, Theorem 3.2] for $p \geq 1$.

PROPOSITION 2.14. Let $X, Y \in \mathbb{B}_A(\mathcal{H})$ and $p \geq 1$. Then

$$\begin{aligned} & 2^{1-4p}w_A^{4p}(X+Y) \\ & \leq \max \left\{ \|X\|_A^{2p}, \|Y\|_A^{2p} \right\} + w_A^p(YX^{*A}) + 2^{2p}w_A^{2p}(X)2w_A^{2p}(Y). \end{aligned}$$

Proof. Assume $x \in \mathcal{H}$ is an A -unit vector. We have

$$\begin{aligned} & 2^{1-4p}|\langle (X+Y)x, x \rangle_A|^{4p} \\ & \leq 2^{1-4p}(|\langle Xx, x \rangle_A| + |\langle Yx, x \rangle_A|)^{4p} \\ & \leq 2^{1-4p}(|\langle x, X^{*A}x \rangle_A| + |\langle x, Y^{*A}x \rangle_A| + 2|\langle Xx, x \rangle_A||\langle Yx, x \rangle_A|)^{2p} \\ & \leq |\langle x, X^{*A}x \rangle_A|^{2p} + |\langle x, Y^{*A}x \rangle_A|^{2p} + 2^{2p}|\langle Xx, x \rangle_A|^{2p}|\langle Yx, x \rangle_A|^{2p} \\ & \quad (\text{by the convexity of } f(t) = t^{2p} \text{ for } p \geq 1) \end{aligned}$$

$$\begin{aligned}
&\leq \max \left\{ \|X^{*A}x\|_A^{2p}, \|Y^{*A}x\|_A^{2p} \right\} + |\langle YX^{*A}x, x \rangle_A|^p + 2^{2p} |\langle Xx, x \rangle_A|^{2p} |\langle Yx, x \rangle_A|^{2p} \\
&\quad (\text{by Lemma 2.12}) \\
&\leq \max \left\{ \|X^{*A}\|_A^{2p}, \|Y^{*A}\|_A^{2p} \right\} + w_A^p(YX^{*A}) + 2^{2p} w_A^{2p}(X) 2w_A^{2p}(Y) \\
&= \max \left\{ \|X\|_A^{2p}, \|Y\|_A^{2p} \right\} + w_A^p(YX^{*A}) + 2^{2p} w_A^{2p}(X) 2w_A^{2p}(Y).
\end{aligned}$$

By taking the supremum on the above inequalities overall A -unit vectors $x \in \mathcal{H}$, we get the desired inequality. $\square$

REFERENCES

1. D. AFRAZ, R. LASHKARIPOUR, AND M. BAKHERAD, Further norm and numerical radius inequalities for sum of Hilbert space operators, *Filomat* **38**(() (2024), 3235–3242.
2. M. AL-DOLAT AND I. JARADAT, A refinement of the Cauchy-Schwarz inequality accompanied by new numerical radius upper bounds, *Filomat* **37**(3) (2023), 971–977.
3. M. AL-DOLAT AND F. KITTANEH, Numerical radius inequalities for certain operator matrices, *J. Anal.* **32**(5) (2024), 2939–2951.
4. M. ALOMARI, S. SAHOO, AND M. BAKHERAD, Further numerical radius inequalities, *J. Math. Inequal.* **16**(1) (2022), 307–326.
5. M.L. ARIAS, G. CORACH, AND M.C. GONZALEZ, Metric properties of projections in semi-Hilbertian spaces, *Integral Equations Operator Theory* **62**(1) (2008), 11–28.
6. M.L. ARIAS, G. CORACH, AND M.C. GONZALEZ, Partial isometries in semi-Hilbertian spaces, *Linear Algebra Appl.* **428**(7) (2008), 1460–1475.
7. H. BAKLOUTI, K. FEKI, AND O.A.M. SID AHMED, Joint numerical ranges of operators in semi-Hilbertian spaces, *Linear Algebra Appl.* **555** (2018), 266–284.
8. W. BANI-DOMI AND F. KITTANEH, Norm equalities and inequalities for operator matrices, *Linear Algebra Appl.* **23**(3) (2020), 1041–1050.
9. P. BHUNIA, Improved bounds for the numerical radius via polar decomposition of operators, *Linear Algebra Appl.* **683** (2024), 31–45.
10. P. BHUNIA, K. FEKI, AND K. PAUL, Numerical radius parallelism and orthogonality of semi-Hilbertian space operators and its applications, *Bull. Iran. Math. Soc.* **47** (2021), 435–457.
11. P. BHUNIA, R.K. NAYAK, AND K. PAUL, Improvement of A -numerical radius inequalities of semi-Hilbertian space operators, *Results Math.* **76**(3) (2021), Paper No. 120.
12. P. BHUNIA, K. PAUL, R.K. NAYAK, On inequalities for A -numerical radius of operator, *Electron. J. Linear Algebra* **36** (2020), 143–157.
13. R.G. DOUGLAS, On majorization, factorization and range inclusion of operators on Hilbert spaces, *Proc. Amer. Math. Soc.* **17** (1966), 413–415.

14. M. EL-HADDAD AND F. KITTANEH, Numerical radius inequalities for Hilbert space operators. II, *Studia Math.* **182**(2) (2007), 133–140.
15. T. FURUTA, J. MIĆIĆ HOT, J. PEČARIĆ, AND Y. SEO, *Mond Pečarić method in operator inequalities*, Element, Zagreb, 2005.
16. M. GUESBA, P. BHUNIA, AND K. PAUL, A -numerical radius inequalities and A -translatable radii of semi-Hilbert space operators, *Filomat* **37**(11) (2023), 3443–3456.
17. M. HAJMOHAMADI, M. LASHKARIPOUR, AND M. BAKHERAD, Further refinements of generalized numerical radius inequalities for Hilbert space operators, *Georgian Math. J.* **28**(1) (2021), 83–92.
18. F. KITTANEH, Numerical radius inequalities for Hilbert space operators, *Studia Math.* **168** (2005), 73–80.
19. F. KITTANEH AND M.H.M. RASHID, Inequalities for norms and numerical radii of operator matrices, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **118**(4) (2024), Paper No. 150.
20. F. KITTANEH AND A. ZAMANI, Bounds for A -numerical radius based on an extension of A -Buzano inequality, *J. Comput. Appl. Math.* **426** (2023), Paper No. 115070.
21. R. LASHKARIPOUR, M. BAKHERAD, AND M. HAJMOHAMADI, Normal shape and numerical range of a real 2-Toeplitz tridiagonal matrix, *Bull. Malays. Math. Sci. Soc.* **46**(3) (2023), Paper No. 102.
22. S. NOURBAKSH, M. HASSANI, M. OMIIDVAR, M. ERFANIAN, AND H.R. MORADI, Inner product inequalities through Cartesian decomposition with applications to numerical radius inequalities, *Oper. Matrices* **18**(1) (2024), 69–81.
23. H. QIAO, G. HAI, AND A. CHEN, Improvements of A -numerical radius for semi-Hilbertian space operators, *J. Math. Inequal.* **18**(2) (2024), 791–810.
24. R. RAHIMI, R. LASHKARIPOUR, AND M. BAKHERAD, Inequalities involving norm and numerical radius of Hilbert space operators, preprint.
25. N. ROUT AND D. MISHRA, Further results on A -numerical radius inequalities, *Ann. Funct. Anal.* **13**(1) (2022), Paper No. 13.
26. M. SABABHEH, S. FURUICHI, AND H.R. MORADI, Operator inequalities via the triangle inequality, *J. Math. Inequal.* **18**(2) (2024), 631–642.
27. S. SHEYBANI, M. SABABHEH, AND H.R. MORADI, Weighted inequalities for the numerical radius, *Vietnam J. Math.* **51**(2) (2023), 363–377.
28. C. YANG, Some generalizations of numerical radius inequalities for 2×2 operator matrices, *J. Math. Inequal.*, to appear.
29. A. ZAMANI, A -numerical radius inequalities for semi-Hilbertian space operators, *Linear Algebra Appl.* **578** (2019), 159–183.

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